Models for integers (in $\mathbb{Z}$)

Dimitris Karlis

Dept of Statistics, Athens University of Economics and Business

April 13, 2024

A starting point

ELSEVIER

Journal of Statistical Planning and Inference 63 (1997) 223–229

journal of statistical planning and inference

Characterizations of a discrete normal distribution

Adrienne W. Kemp

University of St. Andrews, Fife, Scotland KY16 9SS, UK

Received 14 February 1995; received in revised form 14 December 1995

Abstract

The paper obtains a discrete analogue of the normal distribution as the distribution that is characterized by maximum entropy, specified mean and variance, and integer support on $(-\infty, \infty)$. Two alternative characterizations are given, firstly as the distribution of the difference of two related Heine distributions, and secondly as a weighted distribution. © 1997 Elsevier Science B.V.

Keywords: Maximum entropy; Most probable distribution; Discrete normal distribution; Heine distribution; Weighted distribution

Contribution

The maximum entropy principle states that, given partial information about a random variable (rv), it should be modelled using the distribution that satisfies the known constraints and has the maximum entropy

$$H(x) = \sum_{x} p_x \log p_x$$

For a rv $Z \in \mathbb{R}$, then then the normal distribution $N(\mu, \sigma^2)$ is characterized by the property of having maximum entropy for given mean and given variance.
Kemp (1997) showed that a discrete analogue of the normal attains this for rv in $\mathbb{Z}$. The pmf is given by

$$P(X = x) = p_x = \frac{\lambda^x q^{x(x-1)/2}}{\sum_{y=-\infty}^{\infty} \lambda^y q^{y(y-1)/2}}, \quad x = \ldots, -2, -1, 0, 1, 2, \ldots$$

for $\lambda = q^{1/2}$ and $q = \exp(-1/\beta)$ we get a distribution with mean equal to 0 and variance equal to β.

Today

Today:

- Present models/distributions for discrete random variables defined in $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, i.e. the set of all integers including the negative ones.
- There are several mechanisms that lead to such a random variable.
- Define time series models
- Discuss applications

Data in $\mathbb{Z}$

How to obtain such data?

Mostly as the difference of two count variables

- Tick data: price movements in finance
- Score difference in football
- Before and after studies in biostatistics
- Pixel intensity (discrete colors)
- Differencing discrete valued time series to achieve stationarity.

Plan for today

- Define models/distributions in $\mathbb{Z}$
- Extend to the bivariate case
- Consider time series
- Some data application

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Can you name a distribution in $\mathbb{Z}$?

How to define distributions in $\mathbb{Z}$?

- Take the difference of two random variables in $\mathbb{N}$
- Round continuous distributions
- Discrete analogues - discretize
- Random sign

Difference of two discrete distributions in $\mathbb{N}$.

Consider two random variables, say X and Y taking values in $\mathbb{N}$. Then the random variables $Z = X - Y$ will take values in $\mathbb{Z}$ and the probability mass function (pmf) will be given as

$$P(Z = z) = \sum_{k=\max\{0,-z\}}^{\infty} P(X = z + k, Y = k)$$

This implies that given the choice of the distributions we can create a huge number of distributions.
Assumption of independence makes things easier!

Example: Consider two independent rv that follow Poisson distributions and take their difference (nown as Skellam distriubution)

Rounding

- Consider a rv $X \in \mathbb{R}$ (e.g. normal or the Laplace or the logistic etc)
- Consider the integer part of the continuous variable!

Note that this kind of rounding is not unique

If the underlying continuous random variable X has the survival function $S_X(x) = P(X \geq x)$ then the random variable $Z = \lfloor X \rfloor$ denoting the largest integer less or equal to X will have the probability mass function

$$P(Z = z) = P(z \leq X \leq z + 1) = S_X(z) - S_X(z + 1) \ . \quad (1)$$

If X is defined in $\mathbb{R}$ then $Z \in \mathbb{Z}$.

Rounding

- For any given continuous distribution, it is possible to generate corresponding discrete distributions based on (1).
- One can write $X = Z + U$ where U is the fractional part that has been chopped.
- This provides an easy way to derive the moments of the discretized version based on those of the continuous ones. Note that $0 < E(U) < 1$ and $0 < Var(U) < 1/4$.

Discrete analogue of a continuous distribution

Alternatively one may define the discrete analogue of a continuous distribution with density $f(\cdot)$ by considering discretization

$$P(Z = z) = \frac{f(z)}{\sum_{j=-\infty}^{\infty} f(j)} \quad . \tag{2}$$

- This is the derivation of the discrete normal by Kemp(1997)!
- This approach avoids the calculation of the integral involved in the survival function at the cost of deriving the normalizing constant which is an infinite sum.
- In practice this is approximated by a finite sum. For a discussion about such constructions see Chakraborty (2015)

Random Sign

- A different approach is based on assigning a random sign to some discrete variable defined in $\mathbb{N}$.
- Define a random variable Z based on the random variable $X \in \mathbb{N}$ as

$$Z = \begin{cases} X, & \text{with probability } p \\ -X, & \text{with probability } 1-p \end{cases} . \quad (3)$$

- This allows the representation $Z = WX$ where W takes the values 1 and -1 with probabilities p and $1-p$ respectively. (known as Rademacher distribution)
- Extensions assuming that W can take values $-1, 0, 1$ are also possible. (generate zero inflated models)

The Skellam Distribution

If X and Y follow independent Poisson distributions with parameters $\theta_1 > 0$ and $\theta_2 > 0$ respectively, then the random variable $Z = X - Y$ has probability function given by

$$P(Z = z|\theta_1, \theta_2) = e^{-(\theta_1+\theta_2)} \left(\frac{\theta_1}{\theta_2}\right)^{z/2} I_{|z|}\left(2\sqrt{\theta_1\theta_2}\right), \quad z \in \mathbb{Z}, \quad \theta_1, \theta_2 > 0,$$

where $I_r(x)$ is the modified Bessel function of order r

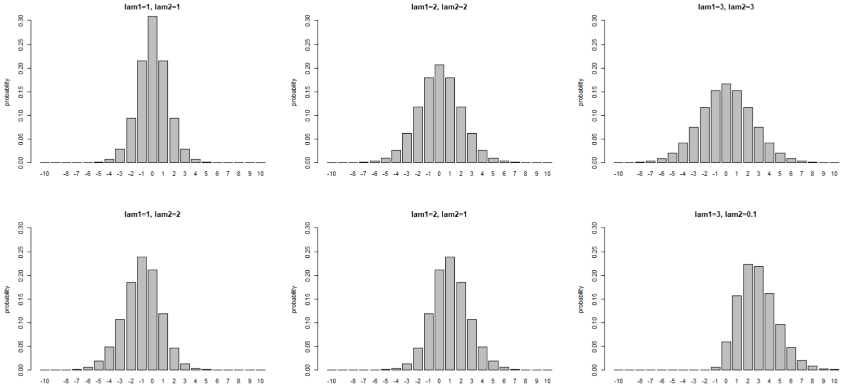
$$I_r(x) = \left(\frac{x}{2}\right)^r \sum_{m=0}^{\infty} \frac{\left(\frac{x^2}{4}\right)^m}{m!\Gamma(r+m+1)}.$$

We will denote this distribution as the Skellam(θ_1, θ_2) distribution.

Properties

- Mean: $\mathrm{E}(Z) = \theta_1 - \theta_2$
- Variance: $\mathrm{Var}(Z) = \theta_1 + \theta_2$,
- For large values of the $\theta_1 + \theta_2$ the distribution can be well approximated by the normal distribution.
- If θ_2 is very close to 0, then the distribution tends to a Poisson distribution.
- Consider two independent random variables $Z_1 \sim \mathrm{Skellam}(\theta_1, \theta_2)$ and $Z_2 \sim \mathrm{Skellam}(\theta_3, \theta_4)$. Then the sum $S_2 = Z_1 + Z_2$ follows a $\mathrm{Skellam}(\theta_1 + \theta_3, \theta_2 + \theta_4)$ distribution, while the difference $D_2 = Z_1 - Z_2$ follows a $\mathrm{Skellam}(\theta_1 + \theta_4, \theta_2 + \theta_3)$ distribution.

Some plots



lam1=1, lam2=1
lam1=2, lam2=2
lam1=3, lam2=3
lam1=1, lam2=2
lam1=2, lam2=1
lam1=3, lam2=0.1
probability
0.00 0.05 0.10 0.15 0.20 0.25 0.30
-10 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10

Properties

- Note: The Skellam distribution is not necessarily that of the difference of two uncorrelated Poisson random variables (Karlis and Ntzoufras (2006)); we can derive the Skellam distribution as the difference of other distributions as well, which motivates its use in various applications.
- To see that, consider X_i, $i = 1, 2, 3$ to be 3 independent variables, with X_1 and X_2 following Poisson distributions, and X_3 following any discrete distribution. Then $X_1 + X_3$ and $X_2 + X_3$ are not independent but their difference, $X_1 - X_2$, follows a Skellam distribution.
- So we can have very many different generating mechanisms

Extensions

- A reparametrized version of the distribution is used, with mean $\mu = \theta_1 - \theta_2$ and variance $\sigma^2 = \theta_1 + \theta_2$. We will denote this by Skellam2(μ, σ^2).
- This allows for better interpretation of the parameters but also more advanced modeling approaches, such as regression.
- Consider

$$\begin{aligned} Y_i &\sim \text{Skellam2}(\mu_i, \sigma^2) \\ \mu_1 &= \beta_0 + \beta_1 X_{i1} + \ldots + \beta_k X_{ik} \end{aligned}$$

to define a Skellam regression model.

Extensions

- Zero inflated version (Karlis and Ntzoufras, 2006) for biostat applications
- Zero deflated version (Koopman et al., 2017). for financial application.
- Truncated Skellam distribution.(Ntzoufras et al., 2021) for volleyball
- Finite mixture of Skellam distribution (Jiang et al., 2014) for clustering in bioinformatics.

Other distributions as difference

- Shahtahmassebi and Moyeed (2016): Generalized Poisson Difference distribution (GPDD)
- Ong et al. (2008) proposed the family of pmfs of the random variable Z of the form $Z = X - Y$ in terms of the Gauss hyper-geometric function ${}_2F_1(\ ;\ ;\ )$, where the random variables X and Y come from the Panjer family of distributions.
- Inusah and Kozubowski (2006): discrete skew discrete Laplace as the difference of two independent geometric variables,
- Bourguignon and Vasconcellos (2016): difference of independent geometric and Poisson variables.
- Chesneau et al. (2022) difference of two independent Poisson-Lindley random variables with the same common parameter.
- Castro (1952): difference between two binomial distributions
- Omair et al. (2016) : difference of two trinomial distributions
- Kemp (1997) showed that the discrete normal can be derived as the difference of two Heine distributions

Other distributions - Discrete Laplace

Kozubowski and Inusah (2006) proposed the discrete skew-Laplace distribution from the continuous skew-Laplace model using discretization with pmf

$$P(Z = z) = \begin{cases} \dfrac{(1-p)(1-q)}{1-pq} q^{|z|}, & z = \ldots, -2, -1 \\ \\ \dfrac{(1-p)(1-q)}{1-pq} p^{k}, & z = 0, 1, 2, \ldots \end{cases} \quad (4)$$

with $p, q \in (0, 1)$.
For $p = q$, the discrete skew-Laplace reduces to the symmetric discrete Laplace in Inusah and Kozubowski (2006) and for either $p = 0$ or $q = 0$, (4) reduces to the geometric distribution.

Discrete Laplace 2 - using rounding

Barbiero (2014) derived a discrete Laplace distribution based on rounding The pmf is now

$$P(Z = z) = \begin{cases} \frac{1}{\log(pq)} \log(p)[q^{-(z+1)}(1-q)] & z = \ldots, -2, -1 \\ \frac{1}{\log(pq)} \log(q)[p^{z}(1-p)] & z = 0, 1, 2, \ldots \end{cases},$$

with $p, q > 0$.

Other

- Chakraborty and Chakravarty (2016): discrete logistic distribution
- Bhati et al. (2020): discrete skew logistic as
- Chakraborty et al. (2021) : discrete Gumbel distribution
- Roy (2003) : discrete normal distribution .
- Ord (1968) defined a discrete Student t-distribution .

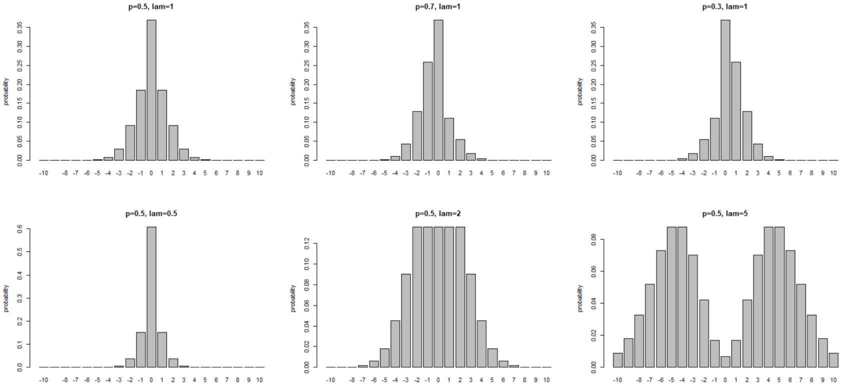
What is the next?

Random Sign

Example: If X follows a Poisson distribution with mean λ and W follows a Rademacher distribution that gives probability p to $W = -1$ and $(1 - p)$ to $W = 1$, then $Z = WX$ follows a signed Poisson distribution (also called an extended Poisson distribution).

- If $p = 0.5$ the distribution is symmetric and has zero mean and variance $\lambda^2 + \lambda$.
- With $p \neq 0.5$ the distribution has mean $\lambda(2p - 1)$ and variance $\lambda^2(4p - 4p^2) + \lambda$.
- For large values of λ the distribution is bimodal.
- For values of λ near 0 the distribution has high probability at 0.
- For certain parameter values the distribution can have a flat mode.

Signed Poisson



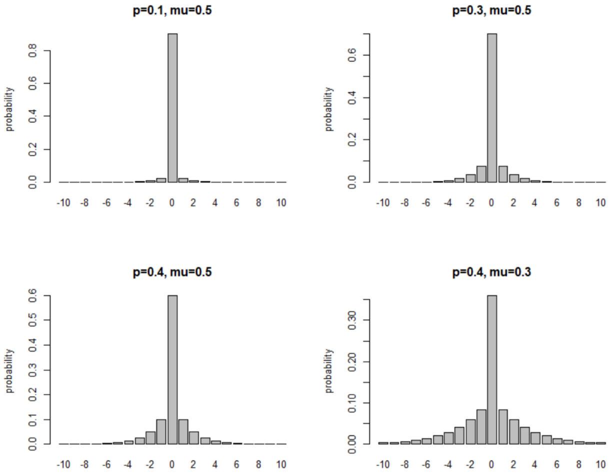
Another one

Xu and Zhu (2022) defined a distribution using $Z = WX$ where W takes values -1,0,1 with probabilities $\rho, 1 - 2\rho$ and ρ respectively and X follows a geometric distribution with pmf

$$P(X = k) = \mu(1 - \mu)^{k-1}, \quad k = 1, 2, \ldots$$

This gives large probability at 0

Signed Geometric



Bivariate Distribution on $\mathbb{Z}^2$

- A bivariate Skellam distribution can be derived using the trivariate reduction method (Bulla et al., 2015).
- Assume that $Y_j \sim \text{Poisson}(\lambda_j)$, independently. Then $Z_1 = Y_1 - Y_0$ and $Z_2 = Y_2 - Y_0$ follow a bivariate Skellam distribution with parameters $\lambda_0, \lambda_1, \lambda_2$ and joint pmf given by

$$\begin{aligned} P(Z_1 = z_1, Z_2 = z_2) &= \exp\left(\lambda_0 + \lambda_1 + \lambda_2\right) \times \\ & \lambda_1^{z_1} \lambda_2^{z_2} \sum_{j=s}^{\infty} \frac{(\lambda_0 \lambda_1 \lambda_2)^j}{(z_1 + j)!(z_2 + j)! j!} \end{aligned}$$

  for all $(z_1, z_2) \in \mathbb{Z}^2$, where $s = \max\{0, -z_1, -z_2\}$.
- For $\lambda_0 = 0$ we get two independent Poisson variates
- for $\lambda_0 > 0$, the covariance of the distribution is given by λ_0.
- The mean and the variance of Z_j is $\lambda_j - \lambda_0$ and $\lambda_j + \lambda_0$ respectively, for $j = 1, 2$.

Bivariate Distribution on $\mathbb{Z}^2$

Genest and Mesfioui (2014) extended this model to some more complex models. Let $\lambda_1 = \min(\lambda_{11}, \lambda_{21}) > 0$ and for fixed $\theta \in [0, \lambda_1]$, let Y_0, Y_1, Y_2 be mutually independent random variables such that

$$\begin{aligned} Y_1 &\sim \text{Skellam}(\lambda_{11} - \theta, \lambda_{12}), \\ Y_2 &\sim \text{Skellam}(\lambda_{21} - \theta, \lambda_{22}), \quad \text{and} \\ Y_0 &\sim \text{Poisson}(\theta), \end{aligned}$$

where $\theta \geq 0$, and with $Y_0 = 0$ if $\theta = 0$.
Then the pair $Z_1 = Y_1 + Y_0, Z_2 = Y_2 + Y_0$ follows a bivariate Skellam distribution of the first kind, since the sum of a Skellam and a Poisson variate is again a Skellam variate.

Bivariate Distribution on $\mathbb{Z}^2$

The second model uses $\lambda_2 = \min(\lambda_{12}, \lambda_{22}) > 0$ and $\Theta = (\theta_1, \theta_2) \in [0, \lambda_1] \times [0, \lambda_2]$. Then let Y_0, Y_1, Y_2 be mutually independent random variables such that

$$\begin{aligned} Y_1 &\sim \text{Skellam}(\lambda_{11} - \theta_1, \lambda_{12} - \theta_2) \quad , \\ Y_2 &\sim \text{Skellam}(\lambda_{21} - \theta_1, \lambda_{22} - \theta_2) \quad \text{and} \\ Y_0 &\sim \text{Poisson}(\theta). \end{aligned}$$

The pair $Z_1 = Y_1 + Y_0, Z_2 = Y_2 + Y_0$ follows a bivariate Skellam distribution of the second kind.

Comments

- Both models allow for only positive correlation.
- To allow for negative correlation Genest and Mesfioui (2014) considered a trivariate reduction of the form $Z_1 = Y_1 + Y_0, Z_2 = Y_2 - Y_0$ which now leads to negative correlation since Y_0 enters with a different sign.
- Properties and estimation about the models are also provided. Further results on estimation of the models are provided in Aissaoui et al. (2017).

Model Based on copula

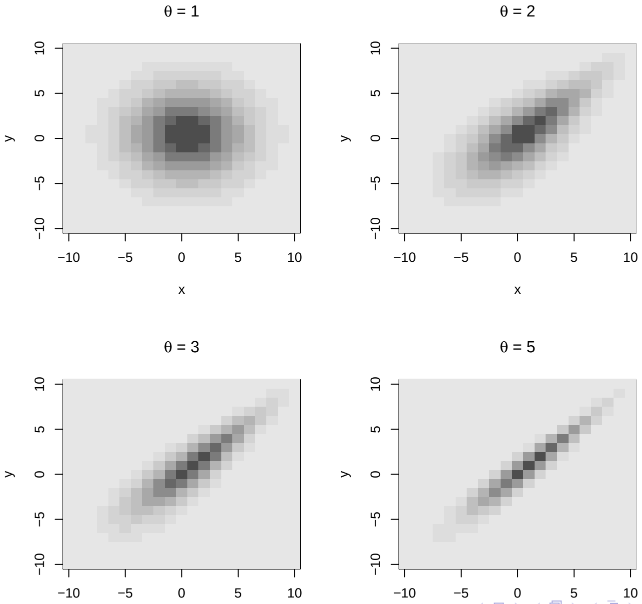
We assume that marginally both of the Y_1 and Y_2 follow a Skellam distribution with parameters $\mu_j, \sigma_j^2,\ j = 1, 2$, coupled with some copula $C(\cdot, \cdot; \theta)$, where θ is the dependence parameter.
For the bivariate case with marginals $F(y_1)$ and $G(y_2)$ one can derive the joint pmf as

$$
\begin{aligned}
P(Y_1 = y_1, Y_2 = y_2) \quad = \quad & C\big(F(y_1), G(y_2); \theta\big) - C\big(F(y_1 - 1), G(y_2); \theta\big) - \\
& C\big(F(y_1), G(y_2 - 1); \theta\big) + C\big(F(y_1 - 1), G(y_2 - 1); \theta\big)
\end{aligned}
$$

Example: Use Gumbel copula

$$
C(u, v; \theta) = \exp\left(-((-\log(u))^{\theta} + (-\log(v))^{\theta})^{1/\theta}\right).
$$

Example



Time Series models

How to extend the AR(1) model

$$Y_t = \phi Y_{t-1} + \epsilon_t$$

to integers in $\mathbb{N}$?

Time Series models

How to extend the AR(1) model

$$Y_t = \phi Y_{t-1} + \epsilon_t$$

to integers in $\mathbb{N}$? Use of some thinning like the binomial thinning

$$Y_t = \alpha \circ Y_{t-1} + R_t$$

- $\alpha \circ X \sim$ Binomial(α, X) or $X = \alpha \circ X = \sum_{i=1}^{X} W_i$ where W_i's are independent Bernoulli rv. (Steutel and Harn, 1979)
- R_t is a sequence of uncorrelated non–negative integer–valued random variables having mean μ and finite variance σ^2 (also called innovations)

INAR models

This is the INAR(1) model with huge extensions (Al-Osh and Alzaid, 1987; McKenzie, 1985)

- different thinning operations
- different innovation distributions
- dependence between thinning and innovations
- many others

How to do it for random variables in $\mathbb{Z}$?

INAR models

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- many others

How to do it for random variables in $\mathbb{Z}$?
Need to define appropriate thinning.

Signed binomial thinning

- The operator $\odot$ is the signed binomial thinning operator defined as (Kim and Park, 2008)

$$\alpha \odot X = \begin{cases} \text{sgn}(\alpha)\text{sgn}(X) \sum_{j=1}^{|X|} W_j(|\alpha|), & X \neq 0 \\ 0, & X = 0 \end{cases} \tag{5}$$

  where $W_j(\alpha)$ is Bernoulli random variable with success probability $|\alpha|$ and $\text{sgn}(X) = 1$ if $X > 0$ and -1 if $X < 0$.
- For $\alpha, X > 0$ the operator coincides with the binomial thinning operator.
- $\alpha \in (-1, 1)$ and is the parameter that relates to the autocorrelation properties of the model.
- Preserves the integer value nature of the process and allows the process to take values in $\mathbb{Z}$.

Signed binomial thinning

The signed binomial thinning operator implies that

$$\alpha \odot X \equiv \begin{cases} Y, & \alpha < 0, \quad X < 0 \\ -Y, & \alpha < 0, \quad X \geq 0 \\ -Y, & \alpha > 0, \quad X < 0 \\ Y, & \alpha > 0, \quad X \geq 0 \end{cases},$$

with $Y \sim Binomial(|\alpha|, |X|)$.
Based on standard properties of binomial random variables we have

$$\begin{aligned} \mathrm{E}(\alpha \odot X \mid X) &= \alpha X \\ \mathrm{Var}(\alpha \odot X \mid X) &= |\alpha|(1 - |\alpha|)|X|. \end{aligned}$$

When $X \geq 0$ and $\alpha \geq 0$, the signed binomial thinning reduces to the binomial thinning.

Signed thinning operator

An earlier version of signed thinning is given in Latour and Truquet (2008). Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. integer-valued random variables with F being their common distribution, independent of an integer-valued random variable X. The signed thinning operator, denoted by $F\circ$, is defined by

$$F \circ X = \begin{cases} \text{sgn}(X) \sum_{i=1}^{|X|} Y_i, & \text{if} \quad X \neq 0 \\ 0, & \text{otherwise} \end{cases} . \tag{6}$$

The sequence $\{Y_i\}_{i=1}^{\infty}$ is referred to as a counting sequence. If the counting sequence Y_i is of Bernoulli type this is related to the signed binomial thinning operator.

More thinning

Recall that the standard INAR model, extends to

$$X_t = S(X_{t-1}) + R_t; \quad t = 0, 1, 2, \ldots \quad , \tag{7}$$

where $S(X)$ is some distribution conditional on X; in the case of binomial thinning operator, a binomial distribution.
Extend to the case of $\mathbb{Z}$

More thinning - not that easy

The operator is defined as

$$S_{\alpha,\theta}(Z) = \mathrm{sgn}(Z) \sum_{i=1}^{|Z|} \xi_i + \sum_{j=1}^{Y(Z)} \eta_i$$

where Z is a $\mathbb{Z}$-valued random variable and $Y(Z)$ follows a Bessel distribution with parameters $|z|$ and θ (see, Devroye, 2002), while ξ_i are Bernoulli random variables with $P(\xi_i = 1) = 1 - P(\xi_i = 0) = \alpha$ independent of η_i, Z and $Y(Z)$. Furthermore η_i is a sequence of i.i.d. random variables independent of Z and Y, with probability mass function $P(\eta_i = 1) = P(\eta_i = -1) = \alpha(1-\alpha)$ and $P(\eta_i = 0) = 1 - 2\alpha(1-\alpha)$, $\alpha \in [0,1]$.
The underlying distribution of $S_{\alpha,\theta}(Z)$ conditional on Z is an extended binomial distribution.

Rounded thinning operator

Kachour and Yao (2009) defined a rounding operator by assuming

$$S(x) = \langle x \rangle$$

where $\langle x \rangle$ stands for the closer integer value to X. In fact this does not assume some distribution, it is deterministic and simply transforms the data to be discrete.

More ...

Let $I(A)$ be the indicator function for A,
$\Delta(X) = \{z \in \mathbb{Z} : z \leq x\}$, $x \in \mathbb{R}$ and

$$B(x) = \frac{[\Delta(x^{1/2}) + 1]^2 - x}{[\Delta(x^{1/2}) + 1]^2 - [\Delta(x^{1/2})]^2},$$

for $x \geq 0$. The operator is defined as

$$\odot(x, U) = \Delta(x) + I\left(U \geq 1 + \Delta(x) - x\right), \quad x \in \mathbb{R}$$

where U is a uniform random variable defined on the interval $[0, 1]$.
The operator returns an integer, selecting between two successive integers with some probability. For example for $x = -4.33$, it will give back -4 with probability 0.67 and -5 with probability 0.33, taking into account the decimal part of the number for this selection probability.

True INAR model (Freeland, 2010).

A new thinning operator, denoted as $\oslash$, which is a kind of binomial thinning operator acting on two latent random variables, can be defined as follows:

$$\alpha \oslash Z_{t-1} | Z_{t-1} = \alpha \circ X_{t-1} - \alpha \circ Y_{t-1} | X_{t-1} - Y_{t-1}.$$

The model is defined as an integer valued stochastic process such that

$$Z_t = \alpha \oslash Z_{t-1} + \epsilon_t, \qquad t = 0, 1, 2, \ldots$$

where ϵ_t is an innovation term defined in $\mathbb{Z}$.

Lot of extensions

Different models generated by considering the difference of two latent processes. Note that $\circ$ refers to binomial thinning, $*$ to negative-binomial thinning and QB to quasi-binomial thinning. X_t and Y_t are the two latent non-negative processes. P stands for Poisson and G for geometric, GP is genetalized Poisson

Latent processes	Reference	X_t	Y_t	Means
$\alpha \circ X_t - \alpha \circ Y_t$	Freeland (2010)	P	P	Same
$\alpha * X_t - \alpha * Y_t$	Nastić et al. (2016)	G	G	Same
$\alpha * X_t - \alpha * Y_t$	Barreto-Souza and Bourguignon (2015)	G	G	Diff
$\alpha * X_t - \beta * Y_t$	Djordjević (2017)	G	G	Diff
$\alpha * X_t - \alpha \circ Y_t$	Bourguignon and Vasconcellos (2016)	G	P	Diff
$QB(X_t) - QB(Y_t)$	Da Cunha et al. (2018)	GP	GP	Diff

INAR type models based on signed binomial thinning

The INAR-model with the signed binomial thinning operator of order p is defined in Kim and Park (2008) as

$$Z_t = \sum_{j=1}^{p} \alpha_j \odot Z_{t-1} + \epsilon_t, \qquad t = 0, 1, 2, \ldots \tag{8}$$

by assuming that the innovations ϵ_t is a sequence of variables in $\mathbb{Z}$ and the operator $\odot$ is the signed binomial thinning operator defined as in (5). The model is called integer-valued autoregressive process of order p with signed binomial thinning (INARS(p)).

More

- Kachour and Truquet (2011) used the signed thinning operator, to derive an order p model The authors avoid, however, a parametric assumption for the innovation term.
- Under a parametric assumption on the common distribution of the counting sequence of the model, Chesneau and Kachour (2012) focused on the parametric SINAR(1) model. They used different choices for the distribution of the counting sequence.
- For example, a natural extension of Bernoulli random variates Y_i to variates in $\mathbb{Z}$ implies that $P(Y_i = -1) = (1-\alpha)^2$, $P(Y_i = 0) = 2\alpha(1-\alpha)$ and $P(Y_i = 1) = \alpha^2$ where $\alpha \in (0,1)$. They also discussed the marginal distribution of the process under different scenarios.

Models based on rounding

Kachour and Yao (2009) defined a model based on a rounding operator. The p-th order model assumes that

$$Z_t = \left\langle \sum_{j=1}^{p} a_j Z_{t-j} + \lambda \right\rangle + \epsilon_t, \qquad t = 0, 1, 2, \ldots$$

where $< \cdot >$ denotes the rounding operator and ϵ_t is a sequence of i.i.d. innovations defined in $\mathbb{Z}$. This model is the Rounded INAR of order p, (RINAR(p)).
The RINAR(p) model is a direct and natural extension on $\mathbb{Z}$ of the AR(p) model for real valued data, for which the rounding operator is a censoring function.

A Skellam INAR model

Let ϵ_t be a sequence of i.i.d. random variables following the Skellam distribution, namely $\epsilon_t \sim \text{Skellam}(\theta_1, \theta_2)$. The PDINAR(1) (Poisson difference of order 1) process Z_t is defined by

$$Z_t = \delta S_{\alpha,\theta}(Z_{t-1}) + \epsilon_t, \qquad t = 0, 1, 2, \ldots$$

where $S_{\alpha,\theta}(Z_{t-1})$ is the extended binomial thinning operator and δ is a parameter with possible values 1 and -1 describing the sign of the correlation (Alzaid and Omair, 2014).

Pergam

Pegram's operator $\star$ is used to mix two (or more) independent discrete random variables U and V over the same sample space $\mathcal{S}$ to produce a new random variable Z.

$$Z = (U, \phi) \star (V, 1 - \phi) \ ;$$

implies that Z takes the value U with probability ϕ and the value V with probability $1 - \phi$. Then the marginal probability of Z is given by $P(Z = j) = \phi P(U = j) + (1 - \phi) P(V = j)$. This can be generalized to the case of k variables,

$$Z = (U_1, \phi_1) \star \ldots \star (U_k, 1 - \sum_{j=1}^{k-1} \phi_j)$$

For a time series in $\mathbb{Z}$ the idea is to construct an order p process as

$$Z_t = (Z_{t-1}, \phi_1) \star \ldots (Z_{t-p}, \phi_p) \star (\epsilon_t, 1 - \phi_1 - \ldots - \phi_p), \qquad t = 0, 1, 2,$$

where ϵ_t is a random variable in $\mathbb{Z}$.

Bivariate models

We consider the extension of the signed thinning operator to the bivariate case.
Let $\mathbf{F}\oplus = \{F_{i,j}\circ\}$ be an 2×2 matrix of signed thinning operators. Let $\mathbf{Z} = (Z_1, Z_2)^T$ be an integer-valued random vector. The effect of $\mathbf{F}\oplus$ on Z, denoted by $\mathbf{F} \oplus \mathbf{Z}$, is defined by

$$\mathbf{F} \oplus \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \oplus \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} F_{11} \circ Z_1 + F_{12} \circ Z_2 \\ F_{21} \circ Z_1 + F_{22} \circ Z_2 \end{pmatrix}$$

This is called the signed matricial-thinning operator, denoted by $\mathbf{F}\oplus$.

Bivariate models

Bulla et al. (2017) defined the B-SINAR(1) (for Bivariate Signed INteger-valued AutoRegressive) process if it admits the following representation

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \mathbf{F} \oplus \begin{pmatrix} Z_{1,t-1} \\ Z_{2,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix}, \qquad t = 0, 1, 2, \ldots$$

where for any $j = 1, 2$, ϵ_{jt} is a sequence of i.i.d integer-valued random variables, with mean μ_j and variance $\sigma^2_{\epsilon_j}$, and independent of all counting sequences of the model. Note that ϵ_{1t} and ϵ_{2t} can be correlated.

Skellam Process

Resembles Brownian motion on the integers.
A Skellam process is defined as

$$Z(t) = N_1(t) - N_2(t), \quad t \geq 0,$$

where $N_1(t)$ and $N_2(t)$, $t \geq 0$ are two independent homogeneous Poisson processes with intensities $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively.
Barndorff-Nielsen et al. (2012) also considered the discrete-Laplace, from the difference of two geometric distributions, and the difference of two negative binomial distributions. The models were applied to finance problems.

Extensions

- Extensions to frcational Skellam process Kerss et al. (2014) Gupta et al. (2020) described a Skellam process of order k, which allows for larger jumps at each step (basketball application)
- Kataria and Khandakar (2021) further extended this to fractional Skellam processes of order k.
- Koopman et al. (2014) proposed a dynamic Skellam model for observations measured over time with serial correlation modeled via stochastically time-varying intensities of the underlying Poisson counts.
- Koopman et al. (2018) extended this to the multivariate

INGARCH models

The Poisson INGARCH(p, q) (Weiß, 2018) is defined as

$$\begin{aligned} X_t \mid \mathcal{F}_{t-1} &\sim \text{Poisson}(\lambda_t) \\ \lambda_t &= \beta_0 + \sum_{i=1}^{p} \beta_i \lambda_{t-i} + \sum_{j=1}^{q} \alpha_j X_{t-j}, \qquad t = 0, 1, 2, \ldots \end{aligned}$$

where $\mathcal{F}_t$ is the available information up to time t, and all α's and β's must be positive.
Extend to the $\mathbb{Z}$

Symmetric Skellam

The first model described in Alomani et al. (2018) assumes that

$$\begin{aligned} Z_t \mid \mathcal{F}_{t-1} &\sim \text{Skellam2}(\mu = 0, \sigma^2) \qquad t = 0, 1, 2, \ldots \\ \sigma^2_{t|t-1} &= \alpha_0 + \alpha_1 Z^2_{t-1} + \beta \sigma^2_{t-1|t-2} \end{aligned}$$

for suitable values of $(\alpha_0, \alpha_1, \beta)$ to keep the variance positive. In this notation $\sigma^2_{t|t-1}$ is the variance of the Skellam at time t conditional on the variance at the point $t-1$.

An asymmetric Skellam

Cui et al. (2021) developed a model based on asymmetric Skellam distribution. The model is defined as

$$
\begin{aligned}
Z_t \mid \mathcal{F}_{t-1} &\sim \text{Skellam}(\mu_{1t}^2, \mu_{2t}^2) \qquad t = 0, 1, 2, \ldots \\
\mu_{1t}^2 &= \alpha_0 + \alpha_1 Z_{t-1}^2 + \beta \mu_{1,t-1}^2, \\
\mu_{2t}^2 &= \alpha_0^* + \alpha_1 Z_{t-1}^2 + \beta \mu_{2,t-1}^2.
\end{aligned}
$$

Note the common parameters α_1 and β. If we add the two parts to create the variance of the Skellam distribution we get

$$\sigma_{t|t-1}^2 = \mu_{1t}^2 + \mu_{2t}^2 = (\alpha_0 + \alpha_0^*) + 2\alpha_1 Z_{t-1}^2 + \beta(\mu_{1,t-1}^2 + \mu_{2,t-1}^2)$$

and thus if $\alpha_0 = \alpha_0^*$ we get the model for the symmetric case of Alomani et al. (2018).

The Generalized Poisson Difference model

Let $X \sim GP(\theta_1, \lambda)$ and $Y \sim GP(\theta_2, \lambda)$, $0 \leq \lambda < 1$, $\theta_1, \theta_2 > 0$ and they are independent. Carallo et al. (2020) proposed a model based on the generalized Poisson difference distribution. For $\lambda = 0$ the two distributions are simple Poisson and we get the Skellam. Using $\mu = \theta_1 - \theta_2$ and $\sigma^2 = \theta_1 + \theta_2$, we denote the distribution $GPDD(\mu, \sigma^2, \lambda)$.

The model assumes that $Z_t \mid \mathcal{F}_{t-1} \sim GPDD(\mu_t, \sigma_t^2, \lambda)$ and then we have

$$\mu_t = \alpha_0 + \sum_{i=1}^{p} \beta_i \mu_{t-i} + \sum_{j=1}^{q} \alpha_j Z_{t-j}$$

for $\alpha_0 \in \mathbb{R}$, $\alpha_i \geq 0$ and $\beta_j \geq 0$, for $i = 1, \ldots, p, j - 1, \ldots, q$ while

$$\sigma_t^2 = |\mu_t|\phi, \quad \phi > (1 - \lambda)^{-2} \quad .$$

Another INGARCH

The model in Hu and Andrews (2021) is slightly different; the observed series is $Z_t = W_t X_t$ where $X_t \mid \mathcal{F}_{t-1} \sim \text{Poisson}(\lambda_t)$ and W_t is a random sign, that takes the value -1 and 1 with equal probability. Then

$$\begin{aligned} \lambda_t &= \frac{\sqrt{1+4\eta_t}-1}{2} \\ \eta_t &= \alpha_0 + \sum_{j=1}^{p} \beta_j \eta_{t-j} + \sum_{i=1}^{q} \alpha_i \left(|Z_{t-i}| - \gamma Z_{t-i}\right)^2 \end{aligned}$$

The marginal distribution of Z_t is not Skellam but rather the signed Poisson distribution

GJR GARCH model

The GJR GARCH model of Xu and Zhu (2022) is defined as with $X_t \mid \mathcal{F}_{t-1}$ following a shifted geometric distribution on $1, 2, \ldots$ with parameter $\phi_t = \{[\rho^2 + 4\rho\eta_t]^{1/2} - \rho\}/\eta_t$. where η_t is defined in (**??**) and W_t is a random variable with

$$P(W_t = k) = \begin{cases} \rho & k = -1, \\ 1 - 2\rho & k = 0, \\ \rho & k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Glosten–Jagannathan–Runkle GARCH (GJR-GARCH) is popular in accounting for asymmetric responses in the volatility

Difference on INGARCH processes

Gonçalves and Mendes-Lopes (2020) considered models for families defined as the difference between two integer valued processes, $Z_t = X_{1t} - X_{2t}$ where X_{it}, $i = 1, 2$ is some integer valued process. They considered examples of geometric GARCH processes with $X_{1t}|X_{1,t-1}$ and $X_{2t}|X_{2,t-1}$ geometrically distributed and analyzed some properties of the process Z_t. The geometric INGARCH process X_t, $t = 0, 1, 2, \ldots$ is a particular case of the negative binomial INGARCH process with

$$P(X_t = k|X_{t-1}) = \frac{1}{1+\lambda_t}\left(1 - \frac{1}{1+\lambda_t}\right)^k, \quad k = 0, 1, 2, \ldots$$

and

$$\lambda_t = \alpha_0 + \sum_{j=1}^{p} \beta_j \lambda_{t-j} + \sum_{i=1}^{q} \alpha_i X_{t-1}.$$

Difference on INGARCH processes

The distribution of $Z_t = X_{1t} - X_{2t}$ is a discrete skew-Laplace

$$P(Z_t = k | X_{1,t-1}, X_{2,t-1}) = \begin{cases} \frac{1}{1+\lambda_{1t}+\lambda_{2t}} \left(\frac{\lambda_{1t}}{1+\lambda_{1t}} \right)^k, & k = 0, 1 \\ \frac{1}{1+\lambda_{1t}+\lambda_{2t}} \left(\frac{\lambda_{2t}}{1+\lambda_{2t}} \right)^{-k}, & k = \ldots \end{cases}$$

which is a reparametrized version of (4). Then they assumed INGARCH dynamics in the difference:

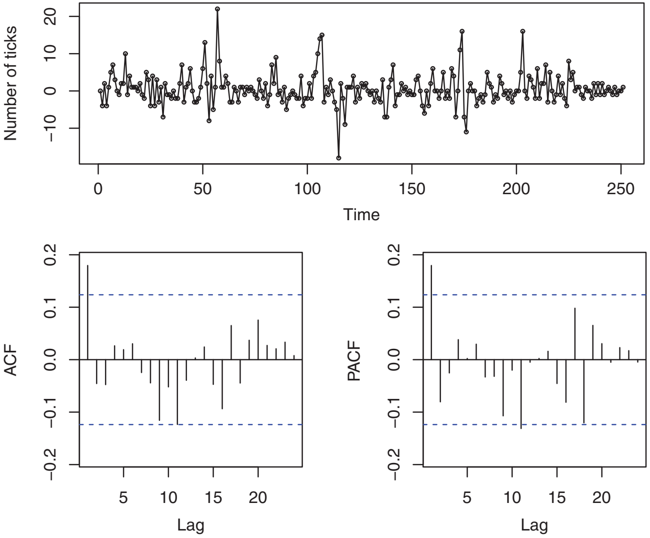
$$\lambda_{jt} = \alpha_0 + \sum_{m=1}^{p} \beta_m \lambda_{j,t-m} + \sum_{i=1}^{q} \alpha_i Z_{j,t-i}$$

for $j = 1, 2$. The model assumes that both integer series and their difference are observed something that perhaps lacks practicality.

Some Data Application

- The data are 251 daily differences between the closing and opening prices of the Saudi Telecom asset in 2012.
- Since these differences are usually measured in tenths of the currency, we work with the rescaled time series (closing price minus opening price) $\times$ 10, which we call number of ticks.
- The number of ticks belongs to $\mathbb{Z}$.

The data



Number of ticks
20
10
0
−10
0
50
100
150
200
250
Time
ACF
0.2
0.1
0.0
−0.1
−0.2
5
10
15
20
Lag
PACF
0.2
0.1
0.0
−0.1
−0.2
5
10
15
20
Lag

The Models

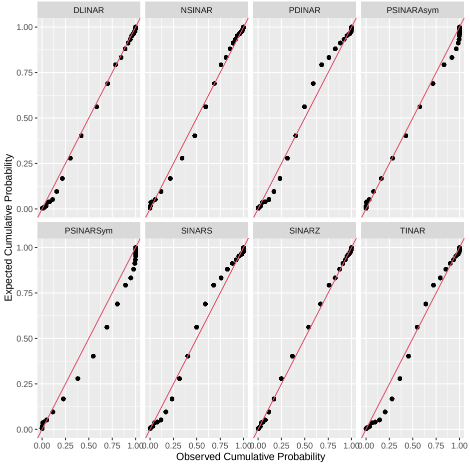
True INAR	Freeland (2010)	TINAR
Discrete-Laplace INAR model	Nastić et al. (2016)	DLINAR
Skew INAR	Barreto-Souza and Bourguignon (2015)	SINARZ
New Skew INAR model	Bourguignon and Vasconcellos (2016)	NSINAR
Poisson Difference	Alzaid and Omair (2014)	PDINAR
Skellam INAR	Andersson and Karlis (2014)	SINARS
Parametric Signed INAR model	Chesneau and Kachour (2012)	PSINARSym
		PSINARAsym

Some info

- Yule-Walker estimation
- Parametric bootstrap standard errors (1000 replications)
- All models are autoregressive of order one with exponentially decaying auto-correlations so they fit the same autocorrelation - compare the marginals

Results

The P-P plots for the stock market data compared to that of all the considered models



Fitted models

Model	$\hat{\alpha}$	Other Parameters		Marginal Distribution	Innovation Distribution	MSE
TINAR	0.1795	$\hat{\lambda}$= 7.4953		Skellam	Skellam	0.0124
	(0.0623)	(0.7886)				
PDINAR	0.1795	$\hat{\theta}_1$=7.6267	$\hat{\theta}_2$7.1950	Skellam	Skellam	0.0128
	(0.0659)	(0.8222)	(0.7988)			
SINARZ	0.1795	$\hat{\theta}_1$=8.7506	$\hat{\theta}_2$=8.3191	Unknown	Skellam	0.0129
	(0.0633)	(0.7991)	(0.7912)			
DLINAR	0.1795	$\hat{\mu}$=2.5635		Discrete Laplace	Not standard	0.0029
	(0.0602)	(0.2192)				
SINARZ	0.1795	$\hat{\mu}_1$=2.6385	$\hat{\mu}_2$= 2.1126	Skew Laplace	Not Standard	0.0031
	(0.0648)	(0.3458)	(0.2757)			
NSINAR	0.1795	$\hat{\mu}$ =3.4262	$\hat{\lambda}$=2.9003	Difference of a geometric and a Poisson	Not Standard	0.0070
	(0.0647)	(0.3720)	(0.2783)			
PSINAR	0.1795	λ=0.9308		Unknown	Symm. Poisson	0.0049
	(0.0647)	(0.1599)				
PSINAR	0.1795	$\hat{p}$= 0.0590	$\hat{\lambda}$=2.9001	Unknown	Asym. Poisson	0.0094
	(0.0709)	(0.0503)	(0.16782)			

Remarks

- A rather new era
- Quite fertile
- Extending results from $\mathbb{R}$ and $\mathbb{N}$.
- Is just discretizing known models sufficient?
- Computational challenges

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