A discrete bivariate distribution and its natural conjugate with applications in bonus-malus systems: the case of independence and dependence

Emilio Gómez–Déniz

Department of Quantitative Methods and TIDES Institute
University of Las Palmas de Gran Canaria
emilio.gomez-deniz@ulpgc.es

Workshop on Discrete Distributions

In Memory of Adrienne Freda Kemp
Saturday April 13, 2024
Harokopio University of Athens

Contents

1. Introduction
2. Basic Distributions
3. The ratemaking system proposed
4. Application
5. Conclussions

- Classical system: the premium assigned to each policyholder is based only on the number of claims made. Therefore, a policyholder who has had an accident producing a relatively small amount of loss is penalised to the same extent as one who has had a more costly accident. This outcome would seem to be unfair.
- We propose a modification of the bonus-malus system of tarification that is commonly applied in automobile insurance by introducing the amount of the claims in a collateral way.
- We also describe a bivariate prior distribution conjugated with respect to the likelihood. This approach produces credibility bonus-malus premiums that satisfy appropriate transition rules. We study the case of independence and dependence between the risk profiles.
- A practical example of its application is presented and the results obtained are compared with those derived from the traditional Poisson-Gamma model in which only the number of claims is taken into account.

The classical Poisson-Gamma model:

$$
\begin{aligned}
f(x|\theta) &= \frac{1}{x!}\theta^x \exp(-\theta), \quad x = 0, 1, \ldots, \quad \theta > 0, \\
\pi(\theta) &= \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}\exp(-\beta\theta),\ \alpha, \beta > 0,\ \theta > 0, \\
P^*(x^+, t) &= \text{Bayes Bonus-Malus Premium} = \frac{\beta}{\alpha}\frac{\alpha + x^+}{\beta + t}, \quad x^+ = \sum_{i=1}^{t} x_i, \\
&= \frac{\beta}{\alpha}\left[\gamma(t)x^+ + [1 - \gamma(t)]\mathbf{E}(\theta)\right],\ \gamma(t) = \frac{\mathbf{E}[var(X|\theta)]}{t\, var[\mathbf{E}(X|\theta)] + \mathbf{E}[var(X|\theta)]}
\end{aligned}
$$

Transition rules:

$$
\frac{\partial P^*(x^+, t)}{\partial x^+} > 0, \quad \frac{\partial P^*(x^+, t)}{\partial t} < 0.
$$

Table 1: Classical model. BMP under Poisson-gamma assumption

	Number of claims				
t	0	1	2	3	4
0	1.000	–	–	–	–
1	0.941	1.750	2.570	3.380	4.190
2	0.889	1.660	2.420	3.190	3.960
3	0.841	1.570	2.290	3.020	3.750
4	0.799	1.490	2.180	2.870	3.560
5	0.761	1.420	2.070	2.730	3.390

We assume that the number of claims follow a Poisson distribution with parameter $\theta > 0$:

$$f(x|\theta) = \frac{1}{x!}\theta^x \exp(-\theta), \quad x = 0, 1, \ldots \tag{1}$$

we include a second random variable, thus giving rise to the consideration of two separate sub-events (claims with size more or less than ψ), as follows. Let Z_i be the variable

$$Z_i = \begin{cases} 1, & \text{if the } i\text{th claim corresponds to a claim size larger than } \psi, \\ 0, & \text{otherwise,} \end{cases}$$

$$f(z_i|p) = \begin{cases} p, & \text{if} \quad z_i = 1, \\ 1-p, & \text{if} \quad z_i = 0, \end{cases}$$

where $0 < p < 1$. Bernoulli distribution.

We now assume that $Z = \sum_{i=1}^{x} Z_i$ is the total number of claims with a claim size larger than ψ. Thus, if the Z_i $(i = 1, \ldots, x)$ are assumed to be mutually independent, then the conditional probability function of Z, given that $X = x$, is binomial with parameters x and p. That is,

$$f(z|x,p) = \binom{x}{z} p^z (1-p)^{x-z}, \quad z = 0, 1, \ldots, x.$$

Therefore, the joint distribution of the number of total claims (X) and the corresponding number of claims with claim size above ψ (Z) has the following probability function:

$$f(x, z|\theta, p) = f(z|x,p) f(x|\theta) = \left(\frac{p}{1-p}\right)^z \frac{(\theta(1-p))^x \exp(-\theta)}{(x-z)!z!},$$

for $x = 0, 1, \ldots,$ $z = 0, 1, \ldots, x$.

$$\rho(X, Z) = \sqrt{p}$$

Prior distributions for risk profiles:

- Gamma distribution:

$$\pi_1(\theta) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} \theta^{\alpha_1 - 1} \exp(-\beta_1 \theta), \quad \theta > 0.$$

- Beta distribution:

$$\pi_2(p) = \frac{1}{B(\alpha_2, \beta_2)} p^{\alpha_2 - 1} (1 - p)^{\beta_2 - 1}, \quad 0 < p < 1,$$

where $B(a, b)$ is the Beta function given by $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ and $\Gamma(\cdot)$ is the Gamma Euler function.

- Joint prior distribution: $\pi(\theta, p) = \pi_1(\theta)\pi_2(p)$.
- Sample data: $(\tilde{x}, \tilde{z}) = \{(x_1, z_1), \ldots, (x_t, z_t)\}$, where t is the sample size.
- Posterior distribution:

$$\pi^*(p_1, p_2|(\tilde{x}, \tilde{z})) \propto p^{\alpha_2^*-1}(1-p)^{\beta_2^*-1}\theta^{\alpha_1^*-1}\exp\left(-\beta_1^*\theta\right),$$

The product of a gamma and a beta with update parameters given by:

$$\begin{aligned} \alpha_1^* &= \alpha_1 + t\bar{x}, \\ \alpha_2^* &= \alpha_2 + t\bar{z}, \\ \beta_1^* &= \beta_1 + t, \\ \beta_2^* &= \beta_2 + t(\bar{x} - \bar{z}). \end{aligned}$$

Here, $\bar{x} = (1/t)\sum_{i=1}^{t} x_i$ y $\bar{z} = (1/t)\sum_{i=1}^{t} z_i$ are the sample mean of X and Z, respectively. $x - z$ is the number of claims corresponding to a size below ψ.

Let $\vartheta = (\theta, p)$ the vector of risk profiles.

- Risk premium, $P(\vartheta)$: value of P which minimizes $\mathbb{E}_f\left[L(g(x,z),P)\right]$, where L is a loss function taken usually as the squared-error loss function.

$$g(x,z) = p_l z + p_s(x-z), \quad p_s < p_l.$$

- Collective premium: value of P which minimizes $\mathbb{E}_{\pi(\vartheta)}\left[L(P(\vartheta),P)\right]$.

- Bayes premium, P^*: value of P which minimizes $\mathbb{E}_{\pi^*(\vartheta|(\tilde{x},\tilde{z}))}\left[L(P(\vartheta),\vartheta)\right]$.

- Risk premium:

$$P(\vartheta) = \sum_{x=0}^{\infty} \sum_{z=0}^{x} g(x,z) f(x,z|\vartheta) = ((p_l - p_s)p + p_s)\theta.$$

If $p_l = p_s = 1$, we get $P(\theta) = \theta$.

- Collective premium

$$P = \int_0^{\infty} \int_0^1 P(\vartheta)\pi(\vartheta)\, d\vartheta = \frac{\alpha_1}{\beta_1} \, \frac{p_l \alpha_2 + p_s \beta_2}{\alpha_2 + \beta_2}.$$

If $p_l = p_s = 1$, $P = \alpha_1 / \beta_1$.

- Bayes premium:

$$P^*(t,x,z) = \frac{\alpha_1^*}{\beta_1^*} \, \frac{p_l \alpha_2^* + p_s \beta_2^*}{\alpha_2^* + \beta_2^*} = \frac{\alpha_1 + x}{\beta_1 + t} \, \frac{p_l(\alpha_2 + z) + p_s(\beta_2 + x - z)}{\alpha_2 + \beta_2 + x},$$

$$x = t\bar{x} = \sum_{i=1}^{t} x_i, \; z = t\bar{z} = \sum_{i=1}^{x} z_i.$$

$P^*(0,0,0) = P$.

$$P^*(t,x,z) = \gamma(t,x)P + [1-\gamma(t,x)]h(x,z,t),$$

where

$$\begin{aligned}\gamma(t,x) &= \frac{\beta_1(\alpha_2+\beta_2)}{(\beta_1+t)(\alpha_2+\beta_2+x)},\\ h(x,z,t) &= \frac{(\alpha_2 p_l + \beta_2 p_s)x + (p_l z + p_s(x-z))(\alpha_1+x)}{(\alpha_2+\beta_2+x)t+\beta_1 x}.\end{aligned}$$

It is easy to see that,

$$\gamma(t,x) = \frac{\mathbb{E}[var(X|\theta)]}{t\, var[\mathbb{E}(X|\theta)] + \mathbb{E}[var(X|\theta)]} \frac{\mathbb{E}[var(Z|p)]}{var[\mathbb{E}(Z|p)] + \mathbb{E}[var(Z|p)]}.$$

Bayes bonus-malus premium,

$$P^{**}(x,z,t) = \frac{P^*(x,z,t)}{P^*(0,0,0)} = \frac{P^*(x,z,t)}{P},$$

Transition rules:

$$\begin{aligned}\frac{\partial P^{**}(x,z,t)}{\partial z} &= \frac{1}{P}\frac{(p_l-p_s)(\alpha_1+x)}{(\beta_1+t)(\alpha_2+\beta_2+x)} > 0,\\ \frac{\partial P^{**}(x,z,t)}{\partial t} &= -\frac{1}{P}\frac{(\alpha_1+x)(p_l(\alpha_2+z)+p_s(\beta_2+x-z))}{(\beta_1+t)^2(\alpha_2+\beta_2+x)} < 0.\end{aligned}$$

We examined a data set based on one-year vehicle insurance policies taken out in 2004 or 2005.

Table 2: Some descriptive data of claims and claim size for the data set

	Number of claims	Size
Mean	0.072	137.27
Stand. Deviation	0.278	1056.30
min	0	0
max	4	55922.10

Table 3: The first 100 observations of the data

1	0	0	21	0	0	41	2	1811.71	61	0	0	81	0	0
2	0	0	22	0	0	42	0	0	62	0	0	82	0	0
3	0	0	23	0	0	43	0	0	63	0	0	83	0	0
4	0	0	24	0	0	44	0	0	64	0	0	84	0	0
5	0	0	25	0	0	45	0	0	65	1	5434.44	85	0	0
6	0	0	26	0	0	46	0	0	66	1	865.79	86	0	0
7	0	0	27	0	0	47	0	0	67	0	0	87	0	0
8	0	0	28	0	0	48	0	0	68	0	0	88	0	0
9	0	0	29	0	0	49	0	0	69	0	0	89	0	0
10	0	0	30	0	0	50	0	0	70	0	0	90	0	0
11	0	0	31	0	0	51	0	0	71	0	0	91	0	0
12	0	0	32	0	0	52	0	0	72	0	0	92	0	0
13	0	0	33	0	0	53	0	0	73	0	0	93	0	0
14	0	0	34	0	0	54	0	0	74	0	0	94	0	0
15	1	669.51	35	0	0	55	0	0	75	0	0	95	0	0
16	0	0	36	0	0	56	0	0	76	0	0	96	1	1105.77
17	1	806.61	37	0	0	57	0	0	77	0	0	97	0	0
18	1	401.80	38	0	0	58	0	0	78	0	0	98	0	0
19	0	0	39	0	0	59	0	0	79	0	0	99	1	200
20	0	0	40	0	0	60	0	0	80	0	0	100	0	0

Table 4: Exact amount corresponding to all the claims made by simulation

2	1811.71	524	1287	2	962.79	446	516	2	2090.86	661	1429	2	1431.5	986	445	2	2087.77	599	1488		
2	739.23	581	158	2	1727.08	364	1363	2	939.74	455	484	2	2717.96	2183	534	2	761.25	121	640		
2	2724.34	1192	1532	2	1677.59	322	1355	2	13277.	12433	843	2	4050.61	1844	2206	2	1262.86	42	1220		
2	19142.1	7726	11416	2	3213.22	827	2386	2	1984.62	417	1567	2	1837.12	1409	428	2	1900.56	1595	305		
2	1043.7	865	178	2	400.	178	222	2	868.78	290	578	2	4987.8	3181	1806	2	3956.13	1870	2086		
2	1255.54	718	537	2	1563.81	1095	468	2	1022.96	830	192	2	962.57	419	543	2	1601.42	102	1499		
2	400.	68	332	2	13116.4	7039	6077	2	2646.14	649	1997	2	3594.25	2720	874	2	2451.52	135	2316		
2	400.01	358	42	2	1713.2	1231	482	2	1464.36	1227	237	2	1088.74	108	980	2	1199.62	1188	11		
2	1642.68	395	1247	2	682.83	492	190	2	3913.37	2255	1658	2	1654.23	1537	117	2	1169.46	46	1123		
2	21769.7	5662	16107	2	848.45	264	584	2	1252.2	298	954	2	2362.38	1965	397	2	1523.19	467	1056		
2	2212.69	529	1683	2	400.	285	115	2	1604.07	538	1066	2	3331.71	1264	2067	2	1185.12	538	647		
2	1875.14	719	1156	2	2010.08	739	1271	2	1231.75	1206	25	2	15992.5	5520	10472	2	1423.02	277	1146		
2	959.42	726	233	2	1895.68	368	1527	2	5563.53	1529	4034	2	9266.2	934	8332	2	7009.42	4583	2426		
2	1987.8	874	1113	2	951.93	137	814	2	3452.17	3369	83	2	1830.95	242	1588	2	7042.26	1588	5454		
2	3608.18	2089	1519	2	427.46	36	391	2	471.74	138	333	2	15066.4	5072	9994	2	2014.36	1607	407		
2	2701.22	2044	657	2	1724.7	696	1028	2	582.69	219	363	2	12335.2	10901	1434	2	2227.05	1259	968		
2	968.33	797	171	2	10675.1	1141	9534	2	400.	259	141	2	9877.79	4065	5812	2	3150.29	2799	351		
2	4108.7	4058	50	2	5997.14	3289	2708	2	3074.81	797	2277	2	4191.34	1337	2854	2	3533.18	2650	883		
2	778.04	14	764	2	567.73	334	233	2	2364.18	266	2098	2	2082.22	1937	145	2	892.86	234	658		
2	553.77	262	291	2	965.95	269	696	2	807.52	640	167	2	1030.66	639	391	2	2416.33	828	1588		
2	5820.54	5287	533	2	10884.1	8503	2381	2	2133.33	1675	458	2	724.82	265	459	2	1409.77	695	714		
2	1394.33	893	501	2	400.	383	17	2	743.08	526	217	2	1375.42	1231	144	2	2135.39	1541	594		
2	4387.48	43	4344	2	2851.75	2729	122	2	19847.7	3642	16205	2	2379.74	1888	491	2	1609.07	529	1080		
2	4419.75	1563	2856	2	2888.81	1616	1272	2	774.76	731	43	2	1750.92	330	1420	2	12463.4	4746	7717		
2	1092.91	74	1018	2	1493.29	391	1102	2	4081.57	781	3300	2	1247.07	1214	33	2	2686.55	676	2010		
2	11653.2	4238	7415	2	1804.99	107	1697	2	3725.	795	2930	2	3184.34	526	2658	2	621.82	467	154		
2	8766.59	8405	361	2	1200.38	74	1126	2	1487.73	962	525	2	2077.22	635	1442	2	680.83	266	414		
2	3974.01	1261	2713	2	3006.17	2065	941	2	1975.41	1205	770	2	822.2	447	375	2	662.86	127	535		
2	662.08	512	150	2	2837.79	1711	1126	2	1278.58	628	650	2	553.8	397	156	2	1194.39	1130	64		
2	3491.53	626	2865	2	2216.65	1067	1149	2	1927.39	1540	387	2	885.91	143	742	2	2187.3	1948	239		
2	7657.57	2873	4784	2	400.	242	158	2	2821.27	151	2670	2	670.05	7	663	2	10333.9	590	9743		
2	830.93	545	285	2	1414.11	1048	366	2	1311.34	479	832	2	9087.83	9029	58	3	4076.82	3290	210	576	
2	710.24	249	461	2	4423.78	2893	1530	2	1480.54	619	861	2	2054.35	1105	949	3	4076.82	3290	210	576	
2	2377.62	1686	691	2	1135.13	941	194	2	1706.88	458	1248	2	3708.5	2352	1356	3	3097.86	554	337	2206	
2	8195.81	3058	5137	2	2926.2	2134	792	2	639.15	175	464	2	1334.51	758	576	3	1231.26	627	360	244	
2	872.59	365	507	2	2297.35	2005	292	2	553.77	328	225	2	593.52	235	358	3	2677.66	2254	195	228	
2	763.97	762	1	2	2763.05	207	2556	2	3965.59	973	2992	2	6607.3	3952	2655	3	2453.27	600	1070	783	
2	508.18	468	40	2	2473.33	10	2463	2	905.23	551	354	2	610.47	435	175	3	1199.67	79	230	890	
2	9511.09	2778	6733	2	801.05	225	576	2	3753.97	1393	2360	2	1275.22	1066	209	3	2716.14	1776	46	894	
2	7474.37	6423	1051	2	783.42	251	532	2	1164.56	1123	41	2	1955.79	1233	722	3	1884.48	913	224	747	
2	400.	262	138	2	1051.13	926	125	2	779.95	672	107	2	1497.7	709	788	3	5026.63	2006	352	2668	
2	6407.89	3051	3356	2	468.91	178	290	2	1771.71	1661	110	2	5608.22	111	5497	3	2351.83	789	63	1499	
2	8458.54	4535	3923	2	1666.06	564	1102	2	718.9	296	422	2	6610.22	4620	1990	3	7797.79	5007	2553	237	
2	8235.63	962	7273	2	4208.6	3494	714	2	1333.4	755	578	2	1871.53	724	1147	3	8990.81	302	3318	5370	
2	1157.85	983	174	2	1317.35	554	763	2	834.99	648	186	2	7297.37	2358	4939	3	10598.4	3212	7066	320	
2	1511.98	245	1266	2	1836.72	1467	369	2	3683.86	1502	2181	2	1303.86	560	743	3	6510.21	779	2683	3048	
2	797.52	778	19	2	2230.74	1532	698	2	890.4	91	799	2	7218.1	3042	4176	3	4054.11	1370	1336	1348	
2	886.1	815	71	2	1355.13	910	445	2	11117.2	10200	917	2	743.77	316	427	3	1353.29	170	748	435	
2	1513.09	1002	511	2	947.57	721	226	2	400.	188	212	2	672.64	631	41	3	5238.85	1007	3844	387	
2	1492.1	407	1085	2	2861.33	1670	1191	2	2056.4	1066	990	2	2398.55	2332	66	3	1173.27	108	359	706	
2	1558.18	316	1242	2	2821.03	1718	1103	2	704.88	293	411	2	2715.72	1380	1335	4	6521.55	652	1304	1956	2608
2	1240.38	1168	72	2	3997.82	2969	1028	2	4939.54	3180	1759	2	3165.	1794	1370	4	2356.41	235	471	706	942
2	3455.58	516	2939	2	1264.06	538	726	2	1618.15	1020	598	2	3731.17	228	3503						
2	1979.68	21	1958	2	850.13	276	574	2	705.25	19	686	2	1657.94	444	1213						
2	400.	130	270	2	1524.31	1009	515	2	1338.9	785	553	2	1426.51	829	597						
2	743.77	564	179	2	581.25	363	218	2	704.44	618	86	2	1132.5	1108	24						
2	1032.02	325	707	2	1180.68	939	241	2	400.003	337	63	2	9359.53	2436	6923						
2	2317.69	1434	883	2	3353.89	3206	147	2	1054.91	798	256	2	2713.27	507	2206						
2	2499.12	2128	371	2	5142.77	2677	2465	2	3321.23	2897	424	2	16667.1	13143	3524						
2	619.09	415	204	2	5108.94	4381	727	2	1428.28	138	1290	2	1565.51	858	707						

Here, we adopt an empirical Bayes approach, which can be viewed as a compromise between Bayes and the classical approach, where the parameters of the prior distributions can be estimated from the data. To do so, we need the marginal (unconditional) distribution of (X, Z), which results

$$f(x,z) = \frac{\beta_1^{\alpha_1} B(\alpha_2 + z, \beta_2 + x - z)}{B(\alpha_2, \beta_2)(\beta_1 + 1)^{\alpha_1 + x}} \binom{x}{z} \binom{\alpha_1 + x - 1}{\alpha_1 - 1},$$

$$cov(X, Z) = \frac{\alpha_1 \alpha_2 (1 + \beta_1)}{\beta_1^2 (\alpha_2 + \beta_2)} > 0.$$

Let $\tau = (\alpha_1, \beta_1, \alpha_2, \beta_2)$ be the vector of the parameters to be estimated and a sample consisting of t observations $(\tilde{x}, \tilde{z}) = \{(\tilde{x}_1, \tilde{z}_1), \dots, (\tilde{x}_t, \tilde{z}_t)\}$, taken from the probability function (2). The log-likelihood is proportional to

$$\begin{aligned} \ell(\tau; (\tilde{x}, \tilde{z})) \quad \propto \quad & t \left[\alpha_1 \log \beta_1 - \log \Gamma(\alpha_1) - \log B(\alpha_2, \beta_2) - (\bar{x} + \alpha_1) \log(1 + \beta_1)\right] \\ & + \sum_{i=1}^{t} \left[\log \Gamma(\alpha_1 + \tilde{x}_i) + \log B(z_i + \alpha_2, \tilde{x}_i - \tilde{z}_i + \beta_2)\right]. \end{aligned}$$

Dependence between the risk profiles

We choose the following prior distribution:

$$\pi(\theta, p) = \pi_1(\theta)\pi_2(p)[1 + \omega\phi_1(\theta)\phi_2(p)], \quad (2)$$

where,

$$\begin{aligned} \phi_1(\theta) &= \exp(-\theta) - \kappa_1, \\ \phi_2(p) &= p - \kappa_2, \\ \kappa_1 &= \left(\frac{\beta_1}{\beta_1 + 1}\right)^{\alpha_1}, \\ \kappa_2 &= \frac{\alpha_2}{\alpha_2 + \beta_2}, \end{aligned}$$

which is a genuine probability density function if $\omega_1 \leq \omega \leq \omega_2$, with

$$\begin{aligned} \omega_1 &= \max\left\{\frac{-1}{\kappa_1\kappa_2}, \frac{-1}{(1-\kappa_1)(1-\kappa_2)}\right\}, \\ \omega_2 &= \min\left\{\frac{1}{\kappa_1(1-\kappa_2)}, \frac{1}{(1-\kappa_1)\kappa_2}\right\}. \end{aligned}$$

$$\pi^*(\theta, p|(\tilde{x}, \tilde{z})) \propto [1 + \omega\phi_1(\theta)\phi_2(p)]p^{\alpha_2^*-1}(1-p)^{\beta_2^*-1}\theta^{\alpha_1^*-1}\exp(-\beta_1^*\theta).$$

Table 5: Observed and fitted values. From top to down, empirical, fitted under independence model and fitted under dependence model

X \ Z	0	1	2	3	4	Total
0	**63232**					**63232**
	63232.50					63232.30
	63233.40					63233.40
1	**1840**	**2493**				**4333**
	1801.81	2526.85				4328.66
	1803.09	2524.36				4327.46
2	**37**	**117**	**117**			**271**
	54.42	121.35	100.73			276.51
	54.67	119.23	102.12			276.72
3	**1**	**5**	**5**	**7**		**18**
	1.76	4.89	6.46	4.12		17.24
	1.78	4.81	6.41	4.28		17.28
4	**0**	**0**	**1**	**0**	**1**	**2**
	0.06	0.19	0.31	0.32	0.17	1.06
	0.06	0.19	0.31	0.32	0.19	1.07
Total	**65110**	**2615**	**123**	**7**	**1**	**67856**
	65090.50	2653.28	107.50	4.44	0.17	67856.00
	65093.00	2648.59	108.84	4.60	0.19	67856.00

Table 6: Parameters estimation, standard errors and some statistics

Parameters	Model: Independent	Model: Dependent
$\widehat{\alpha}_1$	1.154	1.151
	(0.142)	(0.135)
$\widehat{\beta}_1$	15.863	15.820
	(1.964)	(1.879)
$\widehat{\alpha}_2$	5.440	4.833
	(3.538)	(< 0.001)
$\widehat{\beta}_2$	3.879	3.485
	(2.516)	(< 0.001)
$\widehat{\omega}$		–1.847
		(< 0.001)
χ^2_1	5.49	4.61
$g.l.$	2	1
p-value	6.42%	3.20%

Table 7: Independent model. Bayesian bonus–malus premiums proposed for x claims when there are z claims with a claim size larger than ψ and $x-z$ claims with a claim size smaller than ψ. We have assumed that $p_s = 0.5$, $p_l = 1$ and $\psi = 500$

	$x=0$	$x=1$		$x=2$		
t	$z=0$	$z=0$	$z=1$	$z=0$	$z=1$	$z=2$
0	1.000	–	–	–	–	–
1	0.941	1.690	1.800	2.400	2.550	2.690
2	0.888	1.590	1.690	2.270	2.400	2.540
3	0.841	1.510	1.610	2.150	2.280	2.400
4	0.799	1.440	1.530	2.040	2.160	2.280
5	0.760	1.370	1.450	1.940	2.060	2.170

	$x=3$				$x=4$				
t	$z=0$	$z=1$	$z=2$	$z=3$	$z=0$	$z=1$	$z=2$	$z=3$	$z=4$
1	3.080	3.250	3.430	3.600	3.730	3.930	4.130	4.330	4.530
2	2.910	3.070	3.240	3.400	3.530	3.710	3.900	4.090	4.280
3	2.750	2.910	3.060	3.220	3.340	3.520	3.690	3.870	4.050
4	2.620	2.760	2.910	3.060	3.170	3.340	3.510	3.680	3.850
5	2.490	2.630	2.770	2.910	3.020	3.180	3.340	3.500	3.660

Table 8: Dependent model. Bayesian bonus–malus premiums proposed for x claims when there are z claims with a claim size larger than ψ and $x-z$ claims with a claim size smaller than ψ. We have assumed that $p_s = 0.5$, $p_l = 1$ and $\psi = 500$

	$x=0$	$x=1$		$x=2$		
t	$z=0$	$z=0$	$z=1$	$z=0$	$z=1$	$z=2$
0	1.000	–	–	–	–	–
1	0.941	1.680	1.816	2.373	2.551	2.729
2	0.888	1.586	1.714	2.240	2.407	2.575
3	0.841	1.502	1.622	2.122	2.279	2.437
4	0.799	1.426	1.540	2.015	2.163	2.313
5	0.760	1.358	1.465	1.918	2.060	2.201

	$x=3$				$x=4$				
t	$z=0$	$z=1$	$z=2$	$z=3$	$z=0$	$z=1$	$z=2$	$z=3$	$z=4$
1	3.031	3.243	3.455	3.668	3.664	3.904	4.144	4.384	4.625
2	2.862	3.061	3.260	3.460	3.460	3.685	3.911	4.137	4.363
3	2.710	2.898	3.086	3.275	3.277	3.490	3.702	3.916	4.129
4	2.574	2.751	2.927	3.108	3.112	3.313	3.515	3.717	3.919
5	2.451	2.620	2.788	2.957	2.963	3.154	3.346	3.537	3.729

Table 9: Classical model. BMP under Poisson-gamma assumption

	Number of claims				
t	0	1	2	3	4
0	1.000	–	–	–	–
1	0.941	1.750	2.570	3.380	4.190
2	0.889	1.660	2.420	3.190	3.960
3	0.841	1.570	2.290	3.020	3.750
4	0.799	1.490	2.180	2.870	3.560
5	0.761	1.420	2.070	2.730	3.390

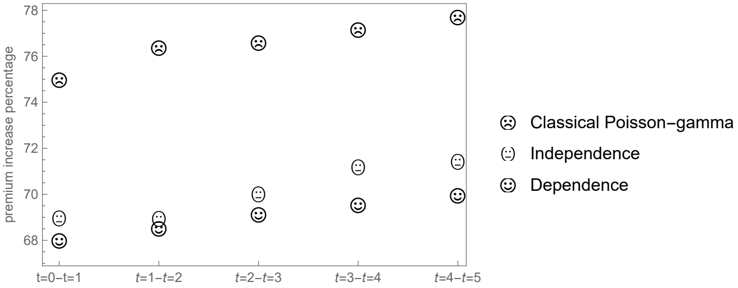

Figure 1: Overcharges

- This work presents a statistical model in which the classical bonus-malus system is modified to distinguish between two different types of claims: those below a limit value and the remainder.

- To do so, we introduce a bivariate distribution obtained under the assumption of dependence. We also describe a bivariate prior distribution which is conjugated with the likelihood of claims being presented. This new approach provides credibility bonus-malus premiums which satisfy desirable transition rules.

- We show that the new model does not modify the discounts made in the absence of claims, but imposes lower penalties (with respect to the negative binomial model) on policyholders who make claims involving amounts below the limit value.

- Other flexible models can be used instead of the model proposed here. For example, the negative binomial-beta model or the Poisson-IG model.

A discrete bivariate distribution and its natural conjugate with applications in bonus-malus systems: the case of independence and dependence

Emilio Gómez–Déniz

Department of Quantitative Methods and TIDES Institute
University of Las Palmas de Gran Canaria
emilio.gomez-deniz@ulpgc.es

Workshop on Discrete Distributions

In Memory of Adrienne Freda Kemp
Saturday April 13, 2024
Harokopio University of Athens