Bivariate Conway-Maxwell-Poisson Distributions

Kimberly F. Sellers[1,2]

Head and Professor
[1]Department of Statistics
North Carolina State University

Principal Researcher
[2]Center for Statistical Research & Methodology
U.S. Census Bureau

Bivariate Poisson Joint PGF (Option 1)

(Marshall & Olkin, 1985; Kocherlakota & Kocherlakota, 1992; Johnson et al., 1997)

- Trivariate reduction:
  - Let $X_i \sim \text{Pois}(\lambda_i), \; i = 1,2,3$
  - Let $X = X_1 + X_3; \;\; Y = X_2 + X_3$

$$\begin{aligned}\Pi(t_1, t_2) &= \exp[\lambda_1(t_1 - 1) + \lambda_2(t_2 - 1) + \lambda_3(t_1 t_2 - 1)] \\ &= \exp\left((\lambda_1 + \lambda_3)(t_1 - 1) + (\lambda_2 + \lambda_3)(t_2 - 1) + \lambda_3(t_1 - 1)(t_2 - 1)\right)\end{aligned}$$

Bivariate Poisson Joint PGF (Option 2)

(Marshall & Olkin, 1985; Kocherlakota & Kocherlakota, 1992; Johnson et al., 1997)

- Compound bivariate binomial with Poisson:
  - Let $n_* \sim \mathrm{Pois}(\lambda_*)$ and $(X, Y|n_*)$ have joint conditional pgf

$$\Pi(t_1, t_2 \mid n_*) = [1 + p_{1+}(t_1 - 1) + p_{+1}(t_2 - 1) + p_{11}(t_1 - 1)(t_2 - 1)]^{n_*}$$

  - Unconditional joint pgf becomes

$$\begin{aligned} \Pi(t_1, t_2) &= \sum_{n_*=0}^{\infty} \frac{\lambda_*^{n_*} e^{-\lambda_*}}{n_*!} \Pi(t_1, t_2 \mid n_*) \\ &= \exp\{\lambda_* p_{1+}(t_1 - 1) + \lambda_* p_{+1}(t_2 - 1) + \lambda_* p_{11}(t_1 - 1)(t_2 - 1)\} \end{aligned}$$

Bivariate Poisson Joint PGF

(Marshall & Olkin, 1985; Kocherlakota & Kocherlakota, 1992; Johnson et al., 1997)

- Via trivariate reduction method:

$$\begin{aligned}\Pi(t_1, t_2) &= \exp[\lambda_1(t_1 - 1) + \lambda_2(t_2 - 1) + \lambda_3(t_1 t_2 - 1)] \\ &= \exp\left((\lambda_1 + \lambda_3)(t_1 - 1) + (\lambda_2 + \lambda_3)(t_2 - 1) + \lambda_3(t_1 - 1)(t_2 - 1)\right)\end{aligned}$$

- Via compounding method:

$$\begin{aligned}\Pi(t_1, t_2) &= \sum_{n_*=0}^{\infty} \frac{\lambda_*^{n_*} e^{-\lambda_*}}{n_*!} \Pi(t_1, t_2 \mid n_*) \\ &= \exp\{\lambda_* p_{1+}(t_1 - 1) + \lambda_* p_{+1}(t_2 - 1) + \lambda_* p_{11}(t_1 - 1)(t_2 - 1)\}\end{aligned}$$

Bivariate Poisson PMF

- The $(X, Y) \sim \text{BivPois}(\lambda_1, \lambda_2, \lambda_3)$ pmf is

$$P(X = x, Y = y) \;=\; \sum_{x_3} \frac{\lambda_1^{x-x_3} \lambda_2^{y-x_3} \lambda_3^{x_3}}{(x - x_3)!(y - x_3)!x_3!} \exp[-(\lambda_1 + \lambda_2 + \lambda_3)]$$

- Form is attained either via trivariate reduction or compounding

Bivariate Poisson: Properties

- Covariance:

$$\mathrm{Cov}(X, Y) = \lambda_3$$

- Correlation:

$$\mathrm{Corr}(X, Y) = \frac{\lambda_3}{\sqrt{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)}} \geq 0$$

- Regression:

$$E(X \mid Y = y) = \lambda_1 + y\left(\frac{\lambda_3}{\lambda_2 + \lambda_3}\right)$$

Problem: Data Dispersion

- Bivariate Poisson assumes equi-dispersion in the random variables, but what if the data don’t satisfy this constraint?
  - If over-dispersed, can consider bivariate negative binomial distribution
  - What if data are under-dispersed?

Bivariate Generalized Poisson Dist.

(Famoye and Consul, 1995)

- Joint pmf:

$$P(X=x, Y=y) = \theta_1\theta_2\theta_3 e^{-\theta_1-\theta_2-\theta_3-x\xi_1-y\xi_2} \sum_{u=0}^{\min(x,y)} k(u)$$

where $k(u) = \frac{[\theta_1+(x-u)\xi_1]^{x-u-1}}{(x-u)!} \frac{[\theta_2+(y-u)\xi_2]^{y-u-1}}{(y-u)!} \frac{[\theta_3+u\xi_3]^{u-1}}{u!} e^{u(\xi_1+\xi_2-\xi_3)}$

- BGPD $\Rightarrow$ BPD when $\xi_1 = \xi_2 = \xi_3 = 0$
- Obtained via trivariate reduction method
  - Let $X_i \sim \mathrm{GP}(\theta_i, \xi_i)$, $i = 1,2,3$
  - Let $X = X_1 + X_3$; $Y = X_2 + X_3$
- Problem when data significantly under-dispersed

Alternative: Consider Bivariate CMP

- Conway-Maxwell-Poisson (COM-Poisson or CMP) distribution is flexible, two-parameter count model allowing for data over-, equi-, or under-dispersion
- Developing a bivariate CMP distribution should likewise capture data over-/under-dispersion

- First, introducing the univariate CMP distribution

COM-Poisson Distribution

(Conway and Maxwell, 1962; Shmueli et al., 2005)

- COM-Poisson (CMP) pmf for rv X:

$$P(X = x) = \frac{\lambda^x}{(x!)^\nu Z(\lambda, \nu)}, \qquad x = 0,1,2, \ldots$$

where $Z(\lambda, \nu) = \sum_{j=0}^{\infty} \frac{\lambda^j}{(j!)^\nu}$; $\quad \nu \geq 0; \quad E(X^\nu) = \lambda$

- Special cases:

Condition(s)	$\boldsymbol{Z(\lambda, \nu)}$ **Simplification**	**Distribution**
$\nu = 1$	e^λ	Poisson(λ)
$\nu = 0, \lambda < 1$	$\frac{1}{1-\lambda}$	Geometric($1 - \lambda$)
$\nu \to \infty$	$1 + \lambda$	Bernoulli$\left(\frac{\lambda}{1+\lambda}\right)$

COM-Poisson Distribution (cont.)

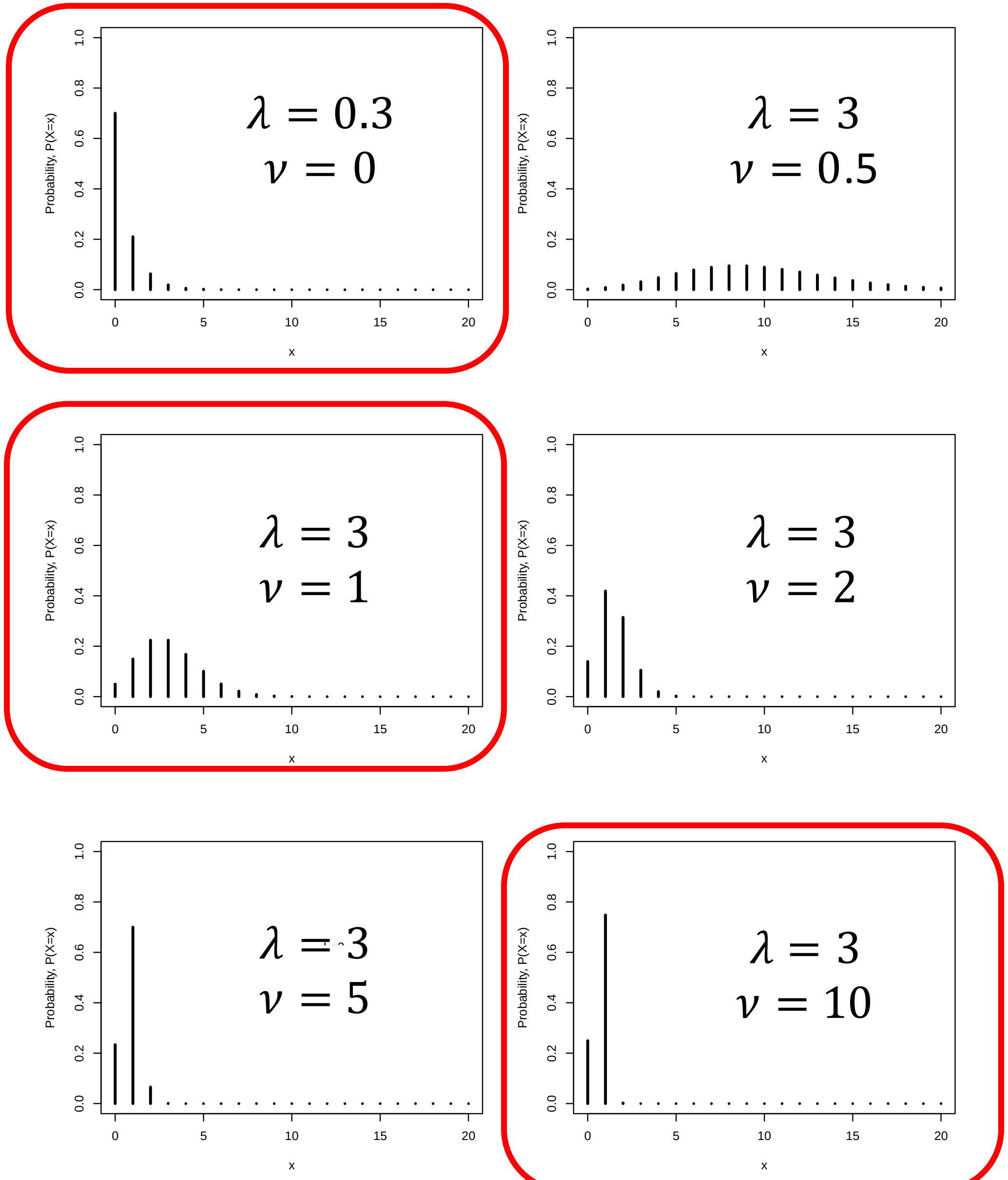
$\text{CMP}(\lambda = 0.3, \nu = 0) =$
$\text{Geom}(p = 1 - \lambda = 0.7)$

- CMP(λ, ν) where $\lambda = 3$, and $\nu \in \{0.5,1,2,5,10\}$
  - First image: $\lambda = 0.3, \nu = 0$

$\text{CMP}(\lambda = 3, \nu = 1) =$
$\text{Poisson}(\lambda = 3)$

- Distribution spread tightens as ν increases

$\text{CMP}(\lambda = 3, \nu \to \infty) =$
$\text{Bern}\left(p = \frac{\lambda}{1+\lambda} = 0.75\right)$



COM-Poisson Distribution Properties

- Has exponential family form

$$\ln L(\lambda, \nu; \boldsymbol{y}) = \sum_{i=1}^{n} y_i \ln \lambda - \nu \sum_{i=1}^{n} \ln(y_i!) - n \ln Z(\lambda, \nu)$$

- Expected value:

$$\mathrm{E}(Y) = \lambda \frac{\partial \ln Z(\lambda, \nu)}{\partial \lambda} \approx \lambda^{1/\nu} - \frac{\nu - 1}{2\nu}$$

- Variance:

$$\mathrm{Var}(Y) = \frac{\partial E(Y)}{\partial \ln \lambda} \approx \frac{1}{\nu} \lambda^{1/\nu}$$

- Approximations hold for $\nu \leq 1$ or $\lambda > 10^{\nu}$

R Computing: CMP Parametrization

Package	Function	Computational result
CompGLM	dcomp	probability mass function
	pcomp	cumulative distribution function
	rcomp	random number generator
compoisson	com.compute.z	normalizing constant, $Z(\lambda, \nu)$
	com.compute.log.z	$\ln(Z(\lambda, \nu))$
	dcom	probability mass function
	com.log.density	log-probability
	com.loglikelihood	log-likelihood
	com.expectation	expectation, $E[f(\cdot)]$
	com.mean	mean
	com.var	variance
COMPoissonReg	dcmp	probability mass function
	pcmp	cumulative distribution function
	qcmp	quantile function
	rcmp	random number generator

Compounded BCMP Distribution

Sellers et al. (2016a)

- Derived via joint conditional bivariate binomial distribution $(X_1, X_2 \mid n)$ where $n \sim \text{CMP}(\lambda, \nu)$ denotes number of trials
- Joint probability:

$$p(x_1, x_2) = \frac{1}{Z(\lambda,\nu)} \sum_{n=0}^{\infty} \frac{\lambda^n}{(n!)^\nu} \times \sum_{a=n-x_1-x_2}^{n} \binom{n}{a, n-a-x_2, n-a-x_1, x_1+x_2+a-n} p_{00}^{a} p_{10}^{n-a-x_2} p_{01}^{n-a-x_1} p_{11}^{x_1+x_2+a-n},$$

- Special cases:
  - Holgate (1964) bivariate Poisson when $\nu = 1$
  - Marshall and Olkin (1985) bivariate Bernoulli when $\nu \to \infty$
  - Bivariate geometric when $\nu = 0, \lambda < 1$
- R computing: multicmp (Sellers et al., 2017a)

Trivariate Reduced BCMP Distribution

Weems et al. (2021)

- Let

$$X_1 = W_1 + W_{12}$$
$$X_2 = W_2 + W_{12}$$

where $W_i \sim \mathrm{CMP}(\lambda_i, \nu)$, and $W_{12} \sim \mathrm{CMP}(\lambda_{12}, \nu)$

- Joint probability:

$$P(X_1 = x_1, X_2 = x_2) = \frac{\lambda_1^{x_1}\lambda_2^{x_2}}{Z(\lambda_1,\nu)Z(\lambda_2,\nu)Z(\lambda_{12},\nu)(x_1!x_2!)^\nu} \times \sum_{j=0}^{\min(x_1,x_2)} \left(\frac{\lambda_{12}}{\lambda_1\lambda_2}\right)^j \left[\binom{x_1}{j}\binom{x_2}{j} j!\right]^\nu$$

- Joint pgf:

$$\Pi(t_1, t_2) = \frac{Z(\lambda_1 t_1,\nu)}{Z(\lambda_1,\nu)} \cdot \frac{Z(\lambda_2 t_2,\nu)}{Z(\lambda_2,\nu)} \cdot \frac{Z(\lambda_{12} t_1 t_2,\nu)}{Z(\lambda_{12},\nu)}$$

Trivariate Reduced BCMP Distribution

Weems et al. (2021)

- Special cases:
  - Holgate (1964) Bivariate Poisson ($\nu = 1$)
  - trivariate reduced bivariate geometric ($\nu = 0$; $\lambda_i < 1$, $i = 1, 2$; $\lambda_{12} < 1$)
  - trivariate reduced bivariate Bernoulli ($\nu \to \infty$)
- Marginal special cases:
  - Poisson($\lambda_i + \lambda_{12}$) when $\nu = 1$
  - sCMP($\lambda, \nu, 2$) when $\lambda_1 = \lambda_2 = \lambda_{12} = \lambda$

*sCMP contains Poisson, binomial, and negative binomial distributions as special cases (Sellers et al., 2017b).

Other BCMP Constructions

- Two Sarmanov constructions (Ong et al., 2021)

$$P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1)P(X_2 = x_2)[1 + \gamma\phi_1(x_1)\phi_2(x_2)], \quad x_j \in \mathbb{R}, \quad j = 1, 2,$$

- First approach based on weighted Poisson distribution framework:

$$\phi_j(x_j) = p^{\alpha}(x_j) - E(p^{\alpha}(X_j)), \quad j = 1, 2,$$

  – Resulting correlation: $\rho = \frac{\gamma(\lambda_1 - \mu_1)(\lambda_2 - \mu_2)}{\sigma_1\sigma_2}$

- Second approach: $\phi_j(x_j) = \theta^{x_j} - \frac{Z(\lambda_j\theta, \nu_j)}{Z(\lambda_j, \nu_j)}, \quad j = 1, 2,$

  – Resulting correlation:

$$\rho = \frac{\gamma\left(\theta\frac{\partial\frac{Z(\lambda_1\theta,\nu_1)}{Z(\lambda_1,\nu_1)}}{\partial\theta} - \mu_1\frac{Z(\lambda_1\theta,\nu_1)}{Z(\lambda_1,\nu_1)}\right)\left(\theta\frac{\partial\frac{Z(\lambda_2\theta,\nu_2)}{Z(\lambda_2,\nu_2)}}{\partial\theta} - \mu_2\frac{Z(\lambda_2\theta,\nu_2)}{Z(\lambda_2,\nu_2)}\right)}{\sigma_1\sigma_2}$$

Other BCMP Constructions

- Copulas (Ötting et al., 2021; Alqawba and Diawara, 2021)

Copula Name	Copula function, $C(u_1, u_2)$	Range for ζ
Ali-Mikhail-Haq	$\frac{u_1 u_2}{1-\zeta(1-u_1)(1-u_2)}$	[-1,1)
Clayton	$\left(\max(u_1^{-\zeta} + u_2^{-\zeta} - 1, 0)\right)^{-1/\zeta}$	$[-1, 0) \cup (0, \infty)$
Frank	$-\frac{1}{\zeta} \ln\left(1 + \frac{(\exp(-\zeta u_1)-1)(\exp(-\zeta u_2)-1)}{\exp(-\zeta)-1}\right)$	$(-\infty, 0) \cup (0, \infty)$
Gaussian	$\Phi_\zeta(\Phi^{-1}(u_1), \Phi^{-1}(u_2))$	[-1,1]
Gumbel	$\exp[-((-\ln(u_1))^\zeta + (-\ln(u_2))^\zeta)^{1/\zeta}]$	$[1, \infty)$
Plackett	$\frac{[1+(\zeta-1)(u_1+u_2)]-\sqrt{[1+(\zeta-1)(u_1+u_2)]^2-4u_1u_2\zeta(\zeta-1)}}{2(\zeta-1)}$	$[0, \infty)$
Reflected Gumbel	$u_1 + u_2 - 1 + \exp[-((-\ln(u_1))^\zeta + (-\ln(u_2))^\zeta)^{1/\zeta}]$	$[1, \infty)$

  – Sklar's theorem: copula function exists such that

$$F(x_1, \dots, x_d) = C\big(F_1(x_1), \dots, F_d(x_d)\big);$$

  in fact, the copula is unique for continuous cdfs

  – Uniqueness property does not apply for discrete (e.g. COM-Poisson) distributions!

Other BCMP Constructions

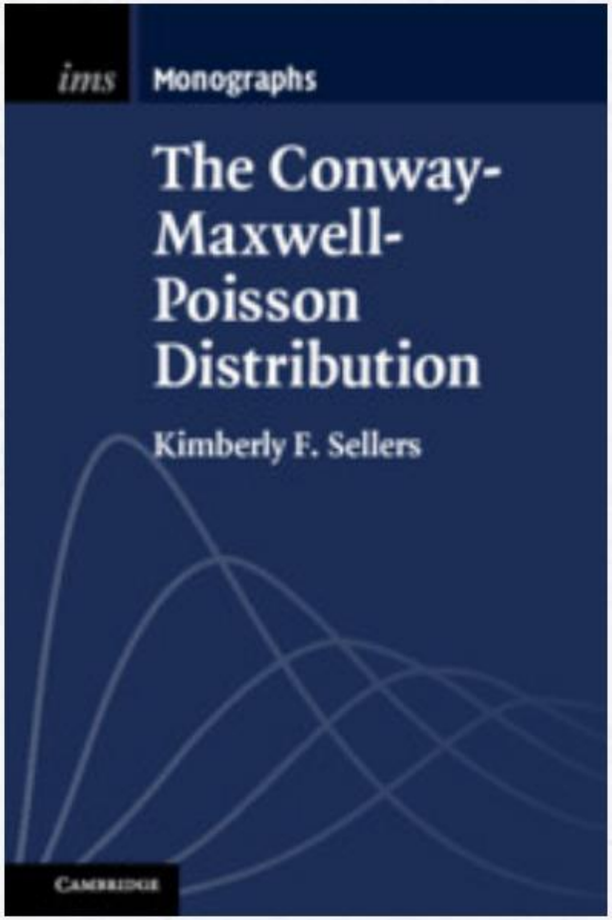
Table 4.6: Bivariate CMP development approaches (trivariate reduction; compounding; the Sarmanov families considering the CMP distribution as a weighted Poisson (Sarmanov 1) or based on the CMP probability generating function (Sarmanov 2), respectively; and copulas) and associated qualities. For each of the considered approaches, the correlation range, and reported special case distributions attainable for the bivariate (Biv.) and marginal (Marg.) distributions are supplied.

	Method				
	Triv. Red.	Compounding	Sarm. 1	Sarm. 2	Copulas
Correlation	[0,1]	[0,1]	[-1,1]	[-1,1]	[-1,1]
Marg. Special Cases	sCMP	Poisson	CMP	CMP	CMP
Biv. Special Cases	Holgate (1964) Poisson	Holgate (1964) Poisson, Bern, Geom	—	Lee (1996) Poisson	See Discussion

Example: Number of NBA All-Star Game Forward and Center players

Model	Parameter MLEs	ln L	No. of param.	AIC	Δ_i
BP	$\hat{\lambda}_1 = 2.941$, $\hat{\lambda}_2 = 2.647$, $\hat{\lambda}_3 = 0$	-54.395	3	114.790	14.269
BNB	$\hat{m} = 2.938$, $\hat{r} = 100000$, $\hat{\alpha}_1 = 0.899$, $\hat{\alpha}_2 = 0$	-54.397	4	116.794	16.273
BGP	$\hat{\theta}_1 = 0.560$, $\hat{\theta}_2 = 0.605$, $\hat{\theta}_3 = 4.048$, $\hat{\lambda}_1 = 0.324$, $\hat{\lambda}_2 = -0.133$, $\hat{\lambda}_3 = -1.000$	-46.661	6	105.322	4.801
BCMPtriv	$\hat{\lambda}_1 = 67.249$, $\hat{\lambda}_2 = 48.573$, $\hat{\lambda}_3 = 0$, $\hat{\nu} = 3.515$	-46.262	4	100.521	$---$
BCMPcomp	$\hat{\lambda} = 1,082,035$, $\hat{\nu} = 8.370$, $\hat{p}_{00} = 0$, $\hat{p}_{01} = 0.158$, $\hat{p}_{10} = 0.185$, $\hat{p}_{11} = 0.658$	-47.986	5	105.972	5.451
BCMPsar1	$\hat{\lambda}_1 = 10.000$, $\hat{\nu}_1 = 2.193$, $\hat{\lambda}_2 = 10.000$, $\hat{\nu}_2 = 2.015$, $\hat{\alpha} = 0.208$, $\hat{\beta} = -1.000$	-46.800	6	105.600	5.079
BCMPsar2	$\hat{\lambda}_1 = 10.000$, $\hat{\nu}_1 = 2.196$, $\hat{\lambda}_2 = 10.000$, $\hat{\nu}_2 = 2.018$, $\hat{\beta} = 1.000$	-48.067	5	106.134	5.613

Learn More About COM-Poisson?



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