On Multivariate Absorption and q-Hypergeometric Distributions

Malvina Vamvakari

April 13, 2024

Workshop on "Discrete Distributions" in Memory of Adrienne Freda Kemp

Outline

q-Preliminaries, $0 < q < 1$

- q-Shifted factorial or q-Pochhammer symbol:

$$(\alpha; q)_n = \prod_{i=1}^{n} \left(1 - \alpha q^{i-1}\right), (\alpha_1, \ldots, \alpha_m; q)_n = (\alpha_1; q)_n \cdots (\alpha_m; q)_n, (\alpha; q)_0 = 1$$

- General q-shifted factorial: $(\alpha; q)_\infty = \prod_{i=1}^{\infty} \left(1 - \alpha q^{i-1}\right)$
- q-Number: $[x]_q = \frac{1-q^x}{1-q}$
- q-Factorial of x of order k: $[x]_{k,q} = [x]_q[x-1]_q \cdots [x-k+1]_q,\ k = 1, 2, \ldots$
- q-Factorial of k: $[k]_q! = [1]_q[2]_q \cdots [k]_q,\ k = 1, 2, \ldots$

- q-Binomial coefficient

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \frac{[n]_{k,q}}{[k]_q!},\ k = 0, 1, \ldots, n$$

- Basic hypergeometric series or q-hypergeometric series

$$_{s+1}\phi_s\left(\begin{array}{c} \alpha_1, \ldots, \alpha_{s+1} \\ b_1, \ldots, b_s \end{array} \middle| q; z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1, \ldots, a_{s+1}; q)_k}{(b_1, \ldots, b_s; q)_k} \frac{z^k}{(q;q)_k}$$

- q-Binomial formula

$$\prod_{i=1}^{n}(1 + tq^{i-1}) = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k}_q t^k = {}_1\phi_0\left(\begin{array}{c} q^{-n} \\ - \end{array} \middle| q; -q^n t\right)$$

- Small q-exponential function

$$e_q(t) = \prod_{i=1}^{\infty}(1 - t(1-q)q^{i-1}t)^{-1} = \sum_{k=0}^{\infty} \frac{t^k}{[k]_q!} = {}_1\phi_0\left(\begin{array}{c} 0 \\ - \end{array} \middle| q; (1-q)t\right)$$

q-Binomial Distribution of the 1^{st} kind

A. Kemp and C. Kemp (1991) defined a q-analogue of the binomial distribution with probability function in the form

$$f_X(x) = \binom{n}{x}_q q^{\binom{x}{2}} \theta^x \prod_{j=1}^{n} (1 + \theta q^{j-1})^{-1}, \; x = 0, 1, \ldots, n,$$

where $\theta > 0, \; 0 < q < 1$.

Heine distribution

A. Kemp and C. Newton (1990) showed that the limit of the pf. of the q-Binomial distribution of the 1^{st} kind, as $n \to \infty$, is the pf. of the *Heine distribution*

$$\lim_{n \to \infty} \binom{n}{x}_q q^{\binom{x}{2}} \theta^x \prod_{j=1}^{n} (1 + \theta q^{j-1})^{-1} = e_q(-\lambda) \frac{q^{\binom{x}{2}} \lambda^x}{[x]_q!}, \, x = 0, 1, \ldots,$$

for $0 < \lambda < \infty$ and $0 < q < 1$, with $\lambda = \theta/(1-q)$

Basic Hypergeometric Series

A. Kemp introduced and studied various forms of discrete q-distributions associated with basic hypegeometric series

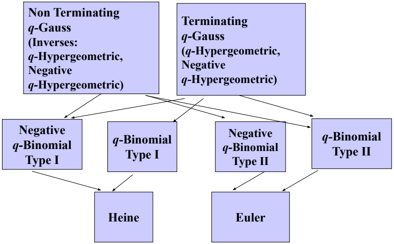
Interpetation

A. Kemp derived discrete q-distributions as stationary distributions of birth and death processes.

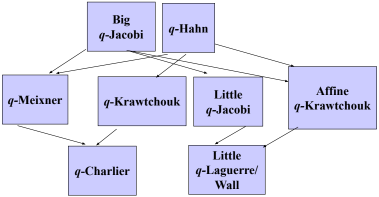
Univariate Discrete q-Distributions (Charalambides, 2016)

Univariate discrete q-distributions are based on stochastic models of sequences of n independent Bernoulli trials with success probability varying geometrically, with rate q, either with the number of previous trials or with the number of previous successes or both with the number of previous trials and successes.

Association with basic orthogonal polynomials (A.Kyriakoussis & M.V., 2010, 2012)

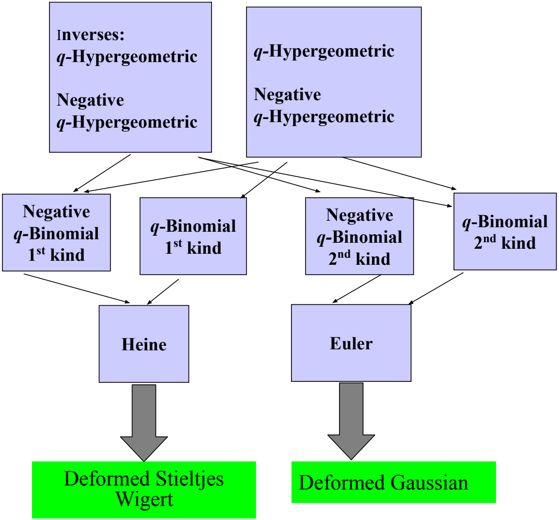


(a) Discrete q-Distributions



(b) Basic Orthogonal Polynomials

Asymptotic Behaviour (Kyriakoussis & M.V., 2013, 2017, 2019)



Univariate Absorption Distribution (A. Kemp(1998), Charalambides (2012, 2016))

Consider a sequence of independent geometric sequences of trials with probability of success at the jth geometric sequence of trials given by

$$p_j = 1 - q^{r-j+1}, \;\; j = 1, 2, \ldots, [r], \;\; 0 < r < \infty, \;\; 0 < q < 1, \qquad (1)$$

which is a geometrically decreasing sequence of a finite number of terms. Then the probability function of the number Y_n of successes in n independent Bernoulli trials is given by

$$f_{Y_n}(y) = P(Y_n = y) = \binom{n}{y}_q q^{(n-y)(r-y)} (1-q)^r [r]_{y,q}, \;\; y = 0, 1, \ldots, n, \qquad (2)$$

for $0 < r < \infty$, $0 < q < 1$, and $n \leq [r]$. This discrete q-distribution is known as *absorption distribution*.

q-Mean, q-Variance

$$\mu_q^A = E\left([Y]_q\right) = (1-q)[n]_q[r]_q,$$

$$\begin{aligned}(\sigma_q^A)^2 &= V\left([Y]_q\right) \\ &= (1-q)^2[n]_{2,q}[r]_{2,q} - (1-q)^2[n]_q^2[r]_q^2 + (1-q)[n]_q[r]_q.\end{aligned}$$

Asymptotic Behaviour of Univariate Absorption Distribution (M.V., 2024)

Theorem

Let $q = q(n)$ *with* $q(n) \to 1$, *as* $n \to \infty$, $q(n)^n = \Omega(1)$ *and* $r = O(n)$. *Then, for* $n \to \infty$, *the univariate absorption distribution is approximated by a deformed standardized Gaussian distribution as follows:*

$$f_Y(y) \cong \frac{(\log q^{-1})^{1/2}}{\sigma_q^A (2\pi(1-q))^{1/2}} \, q^y \exp\left(-\frac{1-q}{2 \log q^{-1}} \left(\frac{[y]_q - \mu_q^A}{\sigma_q^A} \right)^2 \right), \quad y \geq 0.$$

Sketch Proof

- $Z = \frac{[Y]_q - \mu_q^A}{\sigma_q^A}$
- q-Stirling type $0 < q < 1$ (Kyriakoussis and M.V, 2013)

$$[n]_q! = \frac{q^{-1/8}(2\pi(1-q))^{1/2}}{(q\log q^{-1})^{1/2}} \frac{q^{\binom{n}{2}} q^{-n/2} [n]_{1/q}^{n+1/2}}{\prod_{j=1}^{\infty}(1+(q^{-n}-1)q^{j-1})} \left(1 + O(n^{-1})\right),$$

$$[n]_q! = [1]_q[2]_q \cdots [n-1]_q[n]_q \text{ with } [n]_q = \frac{1-q^n}{1-q},\ n \geq 1.$$

- Pointwise convergence techniques applied to the probability function

Remark: Possible realizations of the sequence $q := q(n)$

$$q(n) = 1 - \frac{\alpha}{n}, \quad \alpha > 0 \quad \text{or} \quad q(n) = 1 - 1/\exp n.$$

Univariate q-Hypergeometric Distribution (A. Kemp (2005), Charalambides (2012, 2016), Kyriakoussis & M.V.(2012))

Consider an urn containing r white balls and s white balls. Let W_n be the number of white balls drawn in n q-drawings in a q-hypergeometric urn model, with the conditional probability of drawing a white ball at the q-drawing, given that $j-1$ white balls are drawn in the previous $i-1$ q-drawings given by

$$p_{i,j} = \frac{[r-j+1]_q}{[r+s-i+1]_q}. \tag{3}$$

The distribution of the random variable W_n is called *q-hypergeometric distribution*, with parameters n, r, s and q and its pf. is given by

$$f_{W_n}(w_n) = P(W_n = w) = \binom{n}{w}_q q^{(n-w)(r-w)} \frac{[r]_{w,q}[s]_{n-w,q}}{[r+s]_{n,q}}, \tag{4}$$

for $w = 0, 1, 2, \ldots, n$, where $0 < q < 1$, and r and s are positive integers.

q-Mean, q-Variance

$$
\begin{aligned}
\mu_q^H &= E\left([W_n]_q\right) = \frac{[n]_q[r]_q}{[r+s]_q}, \\
(\sigma_q^H)^2 &= V\left([W_n]_q\right) \\
&= q\frac{[n]_{2,q}[r]_{2,}\,q}{[r+s]_{2,q}} + \frac{[n]_q[r]_q}{[r+s]_q} - \left(\frac{[n]_q[r]_q}{[r+s]_q}\right)^2.
\end{aligned}
$$

Heine Process (Kyriakoussis and M.V., 2017)

Definition

A continuous time process $\{X_q(t), t > 0\}$, is called *Heine process* with parameters q and λ, if the following three assumptions hold
(a) In each time interval of length $\delta = (1-q)t, \ 0 < q < 1,$ for every $t > 0$, at most one event (arrival) occurs with

$$\begin{aligned}\alpha_1(\delta) &= P\left(X_q(t) - X_q(qt) = 1\right) = \frac{\lambda(1-q)t}{1+\lambda(1-q)t},\\ \alpha_0(\delta) &= P\left(X_q(t) - X_q(qt) = 0\right) = \frac{1}{1+\lambda(1-q)t}, \ \lambda > 0.\end{aligned}$$

(b) In the consecutive mutually disjoint time intervals of length $\delta_0 = q^\nu t$ and $\delta_k = (1-q)q^{k-1}t, \ k = 1, 2, \ldots, \nu, \ t > 0, \ \nu \geq 1$, correspond $\nu + 1$ independent events (arrivals).
(c) The process starts at epoch 0 with $X_q(0) = 0$.

Heine Process

- The random variable $X_q(t)$ expresses the number of arivals in the time interval $(0, t]$ and in each time interval of length $\delta_k = (1-q)q^{k-1}$ $k = 1, 2, \ldots,$ occurs one or zero arrival with probabilities

$$\alpha_1(\delta_k) = P\left(X_q(q^{k-1}t) - X_q(q^k t) = 1\right) = \frac{\lambda(1-q)q^{k-1}t}{1+\lambda(1-q)q^{k-1}t}$$

- Heine process has the Heine distribution:

$$P_k(t) = P(X_q(t) = k) = e_q(-\lambda t)\frac{q^{\binom{k}{2}}(\lambda t)^k}{[k]_q!}, \; k = 0, 1, 2, \ldots,$$

for $0 < q < 1$, $0 < \lambda < \infty$.

Theorem (M.V., 2020):On Heine Process and a Multivariate Basic Absorption Distribution

Let the Heine process $\{X_q(t), t > 0\}$ with parameters λ and q. Then, for $t_0 < t_1 < \cdots < t_{\nu-1} < t_\nu = t$, with $t_i = q^{\nu-i}t$, $i = 0, 1, \ldots, \nu$, $\nu \geq 1$, and $0 \leq n_0 \leq n_1 \leq n_2 \leq \cdots n_{\nu-1} \leq n_\nu$, $n_i, i = 0, 1, \ldots, \nu$, nonnegative integers, with $n_0 = k - j$, $0 \leq j \leq k$, and $n_\nu = k$, it holds that

$$P\left(X_q(q^\nu t) = k - j, X_q(q^{\nu-1}t) = n_1, \ldots, X_q(qt) = n_{\nu-1} | X_q(t) = k\right)$$
$$= \binom{k}{x_1, \ldots, x_{\nu-1}, x_\nu}_q q^{(\nu-j)(k-j)} q^{\nu x_1 + (\nu-1)x_2 + \cdots + 2x_{\nu-1} + x_\nu - \binom{j+1}{2}} (1-q)^j,$$

where $x_1 = n_1 - n_0, x_2 = n_2 - n_1, \ldots, x_{\nu-1} = n_{\nu-1} - n_{\nu-2}, x_\nu = k - n_{\nu-1}$, $x_i = 0, 1$, $i = 1, 2, \ldots, \nu, x_0 = n_0 = k-j$ and $x_1+x_2+\cdots+x_{\nu-1}+x_\nu = \sum_{i=1}^{\nu} x_i = j$.

Multivariate Basic Absorpion Disitribution (M.V., 2020)

Definition

Let the discrete νth-variate random variable $(X_1, X_2, \ldots, X_\nu)$ with probability function

$$f_\nu(x_1, x_2, \ldots, x_\nu) = \binom{k}{x_1, x_2, \ldots, x_\nu}_q q^{(\nu - \sum_{i=1}^{\nu} x_i)(k - \sum_{i=1}^{\nu} x_i)} \cdot q^{\sum_{i=1}^{\nu}(\nu - i + 1)x_i - \binom{\sum_{i=1}^{\nu} x_i + 1}{2}} (1-q)^{\sum_{i=1}^{\nu} x_i},$$

where $x_i = 0, 1$, $i = 1, 2, \ldots, \nu$, with $0 \leq \sum_{i=1}^{\nu} x_i \leq k$, k non-negative integer. The distribution of the νth-variate random variable $(X_1, X_2, \ldots, X_{\nu-1}, X_\nu)$ is called *basic multivariate absorption distribution with parameters k and q.*

Multivariate Absorption Distribution (M.V., 2020)

Theorem

Let the discrete νth-variate random vector $\boldsymbol{X} = (X_1, X_2, \dots, X_\nu)$ be distributed according to the basic multivariate absorption distribution. Then the probability function of the m-variate random vector $\boldsymbol{Y} = (Y_1, Y_2, \dots, Y_m)$, with $Y_j = \sum_{i=1}^{\nu_j} X_{\nu - r_j + i}$, $r_j = \sum_{i=j}^{m} \nu_i$, $j = 1, 2, \dots, m$, $\nu = \sum_{i=1}^{m} \nu_i$ and $\nu_i, i = 1 \dots, m$ nonnegative integers, is given by

$$f_{\boldsymbol{Y}}(y_1, y_2, \dots, y_m) = \binom{k}{y_1, y_2, \dots, y_m}_q q^{\sum_{j=1}^{m} y_{j-1}(r_j - z_j)} \prod_{j=1}^{m} (1-q)^{y_j} (\nu_j)_{y_j, q}$$

where $y_j = 0, 1, \dots, k$, $j = 1, 2, \dots, m$, with $\sum_{j=1}^{m} y_j \leq k$, and $z_j = \sum_{i=j}^{m} y_i$, $y_0 = k - z_1$.

Multivariate Absorption Distribution (M.V., 2020)

Definition

Let the m-variate random vector $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$, with probability function

$$f_{\boldsymbol{Y}}(y_1, y_2, \ldots, y_m) = \binom{k}{y_1, y_2, \ldots, y_m}_q q^{\sum_{j=1}^m y_{j-1}(r_j - z_j)} \prod_{j=1}^m (1-q)^{y_j} [\nu_j]_{y_j, q}$$

where $y_j = 0, 1, \ldots, k$, $j = 1, 2, \ldots, m$, with $\sum_{j=1}^m y_j \leq k$, and $z_j = \sum_{i=j}^m y_i$, $y_0 = k - z_1$, $r_j = \sum_{i=j}^m \nu_i$, $j = 1, 2, \ldots, m$, $\nu = \sum_{i=1}^m \nu_i$ and $\nu_i, i = 1 \ldots, m$ nonnegative integers, $k \leq \nu$. The distribution of the m-variate random vector $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_{m-1}, Y_m)$ is called *multivariate absorption distribution with parameters k and q*.

Multivariate q-Hypergeonetric Distribution (M.V., 2020)

Corollary

Let the discrete $\nu-$variate random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_\nu)$ be distributed according to the basic multivaviate absorption distribution and consider the m-vaviate random vector $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$, with $Y_j = \sum_{i=1}^{\nu_j} X_{\nu-r_j+i}$, $r_j = \sum_{i=j}^{m} \nu_i$, $j = 1, 2, \ldots, m$. Then the conditional probability function of $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$, given $Y_0 = k - \sum_{j=1}^{m} Y_j$, is given by

$$f_{\boldsymbol{Y}|Y_0}((y_1, y_2, \ldots, y_{m-1})|y_0) = \binom{k-y_0}{y_1, y_2, \ldots, y_{m-1}}_q q^{\sum_{j=1}^{m-1} y_j(r_{j+1}-z_{j+1})} \frac{\prod_{j=1}^{m}[\nu_j]_{y_j,q}}{[\nu]_{k-y_0,q}},$$

where $y_j = 0, 1, 2, \ldots, k$, $j = 1, 2, \ldots, m-1$, with $\sum_{j=1}^{m-1} y_j \leq k - y_0$, and $z_j = \sum_{i=j}^{m} y_i$, $y_m = k - y_0 - \sum_{j=1}^{m-1} y_j$.

Multivariate q-Hypergeometric Distribution (M.V., 2020)

Definition

Let a discrete $(m-1)$-variate random variable $\boldsymbol{W} = (W_1, W_2, \ldots, W_{m-1})$ with joint probability function

$$f_{\boldsymbol{W}}(w_1, w_2, \ldots, w_{m-1}) = \binom{n}{w_1, w_2, \ldots, w_{m-1}}_q q^{\sum_{j=1}^{m-1} w_j (r_{j+1} - s_{j+1})} \frac{\prod_{j=1}^{m} [\nu_j]_{w_j,q}}{[\nu]_{n,q}},$$

where $w_m = n - \sum_{j=1}^{m-1} w_j$, $w_j = 0, 1, \ldots, n$, $n \geq 0$, $j = 1, 2, \ldots, m-1$, with $\sum_{j=1}^{m-1} w_j \leq n$ and $\nu = \sum_{j=1}^{m} \nu_j$ with $\nu_1, \nu_2, \ldots, \nu_m$ nonnegative integers; also it is set $r_j = \sum_{i=j}^{m} \nu_i$ and $s_j = \sum_{i=j}^{m} w_i$. The distribution of the multivariate discrete random variable $(W_1, W_2, \ldots, W_{m-1})$ is called *multivariate q-Hypergeometric distribution with parameters* $n, \nu_1, \nu_2, \ldots, \nu_m$ *and* q.

Multivariate Discrete q-Distributions (Charalambides, 2021, 2022, 2023)

Multivariate discrete q-distributions are based on stochastic models of sequences of n independent Bernoulli trials with chain-composite successes, where the odds of success of a certain kind at a trial is assumed to vary geometrically, with rate q, with the number of previous trials or with the number of previous successes or both with the number of previous trials and successes.

Multivariate Absorption Distribution (Charalambides, 2022)

Joint probability function of the random vector $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_k)$:

$$f_{\mathcal{Y}}(y_1, y_2, \ldots, y_k) = P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k)$$
$$= \binom{n}{y_1, y_2, \ldots, y_k}_q q^{\sum_{j=1}^{k}(n-s_j)(m_j-y_j)} \prod_{j=1}^{k}(1-q)^{y_j}[m_j]_{y_j,q},$$

$y_j = 0, 1, 2, \ldots, n$, $\sum_{j=1}^{k} y_j \leq n$, $s_j = \sum_{i=1}^{j} y_i$, $0 < m_j < \infty$, $0 < q < 1$, $n \leq [m_j]$, $j = 1, 2, \ldots, k$.

Multivariate q-Hypergeometric Distribution (Charalambides, 2022)

Joint probability function of a random vector $\mathcal{W} = (W_1, W_2, \ldots, W_k)$:

$$f_{\mathcal{W}}(w_1, w_2, \ldots, w_k) = P(W_1 = w_1, W_2 = w_2, \ldots, W_k = w_k)$$
$$= \binom{n}{w_1, w_2, \ldots, w_k}_q q^{\sum_{j=1}^{k}(n-s_j)(\nu_j - w_j)} \frac{\prod_{j=1}^{k+1}[\nu_j]_{w_j,q}}{[\nu]_{n,q}}$$

$w_j = 0, 1, 2, \ldots, n$, $j = 0, 1, 2, \ldots, k$, with $\sum_{j=1}^{k} n_j \leq n$, where $w_k = n - \sum_{j=1}^{k} w_j, \nu = \sum_{j=1}^{k+1} \nu_j$, $s_j = \sum_{i=1}^{j} w_i$, $0 < q < 1$, and $n \leq [\nu_j]$, $j = 1, 2, \ldots, k$.

Asymptotic Behaviour of Bivariate Absorption Distribution (M.V., 2024)

Let the discrete bivariate random variable (Y_1, Y_2) with joint probability function

$$f_{Y_1,Y_2}(y_1, y_2) = \binom{n}{y_1, y_2}_q (1-q)^{y_1+y_2} q^{(\nu-y_1-y_2)(n-y_1-y_2)} q^{y_1(\nu_2-y_2)} [\nu_1]_{y_1,q} [\nu_2]_{y_2,q},$$

where $y_j = 1, 2, \ldots, n$, $j = 1, 2$, with $y_1 + y_2 \leq n$, and $\nu = \nu_1 + \nu_2$, ν_1, ν_2 nonnegative integers.

Marginal probability function of Y_2: Univariate Absorption

The marginal probability function of the random variable Y_2, is distributed according to the univariate absorption distribution with probability function

$$f_{Y_2}(y_2) = \binom{n}{y_2}_q (1-q)^{y_2} q^{(\nu_2-y_2)(n-y_2)} [\nu_2]_{y_2,q}, \quad y_2 = 0, 1, 2, \ldots, n,$$

for $0 < q < 1$ and $n \leq \nu_2$.

q-Mean, q-Variance

$$\mu_{[Y_2]_q} = E\left([Y_2]_q\right) = (1-q)[n]_q[\nu_2]_q$$

$$\begin{aligned}(\sigma_{[Y_2]_q})^2 &= V\left([Y_2]_q\right) \\ &= (1-q)^2[n]_{2,q}[\nu_2]_{2,q} - (1-q)^2[n]_q^2[\nu_2]_q^2 + (1-q)[n]_q[\nu_2]_q\end{aligned}$$

Marginal probability function of Y_1: Not Univariate Absorption

- q-mean and q variance of random variable Y_1 cannot be inferred from the corresponding q-moments of the univariate absorption distribution
- q-mean and q-variance cannot be found explicitly either directly or indirectly

Distribution of the conditional random variable $Y_1|Y_2$: Univariate Absorption

The conditional random variable $Y_1|Y_2$, is distributed according to the univariate absorption distribution with probability function

$$f_{Y_1|Y_2}(y_1|y_2) = \binom{n-y_2}{y_1}_q (1-q)^{y_1} q^{(\nu_1-y_1)(n-y_1-y_2)} [\nu_1]_{y_1,q},$$

$y_1 = 0, 1, 2, \ldots, n-y_2$, $0 < q < 1$, $n - y_2 \leq \nu_1$.

Conditional q-Mean, q-Variance

Conditional mean and conditional variance of the deformed variable $[Y_1]_q$ given $Y_2 = y_2$:

$$\mu_{[Y_1]_q|Y_2} = E\left([Y_1]_q|y_2\right) = (1-q)[n-y_2]_q[\nu_1]_q,$$

$$\begin{aligned}(\sigma_{[Y_1]_q|Y_2})^2 &= V\left([Y_1]_q|y_2\right)\\ &= (1-q)^2\left([n-y_2]_{2,q}[\nu_1]_{2,q} - [n-y_2]_q^2[\nu_1]_q^2\right)\\ &\quad +(1-q)[n-y_2]_q[\nu_2]_q.\end{aligned}$$

Note

Conditional q-Mean: q-Regression Curve

Asymptotic Behaviour of Bivariate Absorption Distribution (M.V., 2024)

Theorem

Let $q = q(n)$ with $q(n) \to 1$, as $n \to \infty$, $q(n)^n = \Omega(1)$ and $\nu_i = O(n)$, $i = 1, 2$. Then, for $n \to \infty$, the bivariate absorption distribution is approximated by a deformed bivariate standardized Gaussian distribution as follows:

$$f_{Y_1,Y_2}(y_1, y_2) \cong \frac{\log q^{-1}}{2\pi(1-q)\sigma_{[Y_2]_q}\sigma_{[Y_1]_q|Y_2}} q^{y_1+y_2}$$
$$\cdot \exp\left(-\frac{1-q}{2\log q^{-1}} \cdot \left(\left(\frac{[y_2]_q - \mu_{[Y_2]_q}}{\sigma_{[Y_2]_q}}\right)^2 + \left(\frac{[y_1]_q - \mu_{[Y_1]_q|Y_2}}{\sigma_{[Y_1]_q|Y_2}}\right)^2\right)\right),$$
$$y_1, y_2 \geq 0.$$

Sketch Proof

- $Z = \frac{[Y_2]_q - \mu_{[Y_2]_q}}{\sigma_{[Y_2]_q}}$
- $W = \frac{[Y_1]_q - \mu_{[Y_1]_q|Y_2}}{\sigma_{[Y_1]_q|Y_2}}$
- q-Stirling type
- Pointwise convergence techniques applied to the joint probability function

Asymptotic Behaviour of Multivariate Absorption Disitribution (M.V., 2020)

Marginal probability function of Y_k: Univariate Absorption

Marginal probability functions of Y_i, $i = 1, \ldots, k-1$, $k \geq 2$: Not Univariate Absorptions

q-means and q-variances of random variables Y_i, $i = 1, \ldots, k-1$, $k \geq 2$ cannot be found

Distributions of the conditional r.vs $Y_{k-1}|Y_k, Y_{k-2}|(Y_{k-1}, Y_k), \ldots, Y_1|(Y_2, \ldots, Y_k)$: Univariate Absorptions

Conditional q-means, q-variances:

$$\mu_{[Y_{k-1}]_q|Y_k}, \mu_{[Y_{k-2}]_q|(Y_{k-1},Y_k)}, \cdots, \mu_{[Y_1]_q|(Y_2,\ldots,Y_k)},$$

$$\sigma^2_{[Y_{k-1}]_q|Y_k}, \sigma^2_{[Y_{k-2}]_q|(Y_{k-1},Y_k)}, \cdots, \sigma^2_{[Y_1]_q|(Y_2,\ldots,Y_k)}$$

The mean and the variance of the deformed variable $[Y_k]_q$ are given by

$$\begin{aligned}\mu_{[Y_k]_q} &= E\left([Y_k]_q\right) = (1-q)[n]_q[m_k]_q \\ &\text{and} \\ (\sigma_{[Y_k]_q})^2 &= V\left([Y_k]_q\right) \\ &= q(1-q)^2[n]_{k,q}[m_k]_{2,q} \\ &\quad -(1-q)^2[n]_q^2[m_k]_q^2 + (1-q)[n]_q[m_k]_q,\end{aligned} \tag{5}$$

respectively.

The conditional mean and the conditional variance of the deformed variable $[Y_{k-1}]_q$ given $Y_k = y_k$ are given by

$$\begin{aligned}\mu_{[Y_{k-1}]_q|Y_k} &= E\left([Y_{k-1}]_q|y_k\right) = (1-q)[n-y_k]_q[m_{k-1}]_q \\ &\text{and} \\ (\sigma_{[Y_{k-1}]_q|Y_{n,k}})^2 &= V\left([Y_{k-1}]_q|y_k\right) \\ &= q(1-q)^2[n-y_k]_{2,q}[m_{k-1}]_{2,q} \\ &\quad -(1-q)^2[n-y_k]_q^2[m_{k-1}]_q^2 + (1-q)[n-y_k]_q[m_{k-1}]_q,\end{aligned}$$

respectively.

The conditional mean and conditional variance of the deformed variables $[Y_j]_q$ given $Y_{j+1} = y_{j+1}, \ldots, y_{n,k} = y_k$, $j = 1, \ldots, k-1,\ \ k \geq 2$, are given respectively by

$$\mu_{[Y_j]_q|\left(Y_{j+1},Y_{j+2},\ldots,Y_k\right)} = E\left([Y_j]_q|\left(y_{j+1}, y_{j+2}, \ldots, Y_k\right)\right)$$

$$= (1-q)\left[n - \sum_{i=j+1}^{k} y_i\right]_q [m_j]_q,$$

$$\sigma^2_{[Y_j]_q|\left(Y_{j+1},Y_{j+2},\ldots,Y_k\right)} = V\left([Y_j]_q|\left(y_{j+1}, y_{j+2}, \ldots, Y_k\right)\right)$$

$$= q(1-q)^2\left[n - \sum_{i=j+1}^{k} y_i\right]_{2,q} [m_j]_{2,q} - (1-q)^2\left[n - \sum_{i=j+1}^{k} y_i\right]_q^2 [m_j]_q^2$$

$$+(1-q)\left[n - \sum_{i=j+1}^{k} y_i\right]_q [m_j]_q.$$

Note

The conditional q-means, $\mu_{[Y_j]_q|(Y_{j+1},Y_{n,j+2},\ldots,Y_k)}$, $1 \leq j \leq k-2$, $k \geq 3$, can be interpreted as q-regression hyperplanes.

Asymptotic Behaviour of Multivariate Absorption Distribution

Theorem

Let $q = q(n)$ *with* $q(n) \to 1$*, as* $n \to \infty$*,* $q(n)^n = \Omega(1)$ *and* $m_j = O(n), j = 1, 2, \ldots, k$*. Then, for* $n \to \infty$*, the multivariate absorption distribution is approximated by a deformed multivariate standardized Gaussian distribution as follows:*

$$f_{\mathcal{Y}}(y_1, y_2, \ldots, y_k) \cong \left(\frac{\log q^{-1}}{2\pi(1-q)}\right)^{k/2} \frac{q^{\sum_{j=1}^{k} y_j}}{\sigma_{[Y_k]_q} \prod_{j=2}^{k} \sigma_{[Y_{j-1}]_q|(Y_j,\ldots,Y_k)}}$$
$$\cdot \exp\left(\frac{1-q}{2\log q}\left(\left(\frac{[y_k]_q - \mu_{[Y_k]_q}}{\sigma_{[Y_k]_q}}\right)^2 + \sum_{j=2}^{k}\left(\frac{[y_{j-1}]_q - \mu_{[Y_{j-1}]_q|(Y_j,\ldots,Y_k)}}{\sigma_{[Y_{j-1}]_q|(Y_j,\ldots,Y_k)}}\right)^2\right)\right),$$
$$y_j \geq 0, j = 1, 2, \ldots, k.$$

Sketch Proof

- $Z_k = \frac{[Y_k]_q - \mu_{[Y_k]_q}}{\sigma_{[Y_k]_q}}$,
- $Z_j = \frac{[Y_j]_q - \mu_{[Y_j]_q | \left(Y_{j+1}, Y_{j+2}, \ldots, Y_k\right)}}{\sigma_{[Y_j]_q | \left(Y_{j+1}, Y_{nj+2}, \ldots, Y_k\right)}}, \ j = 1, \ldots, k-1, \ k \geq 2$
- q-Stirling type
- Pointwise convergence techniques applied to the joint probability function

Asymptotic Behaviour of Multivariate q-Hypergeometric Disitribution

Theorem

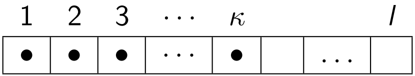
Let $q = q(n)$ with $q(n) \to 1$, as $n \to \infty$, $q(n)^n = \Omega(1)$ and $\nu = O(n)$. Then, for $n \to \infty$, the multivariate q-Hypergeometric distribution is approximated by a deformed multivariate standardized Gaussian distribution as follows:

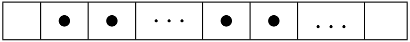
$$f_{\mathcal{W}}(w_1, w_2, \ldots, w_k) \cong \left(\frac{\log q^{-1}}{2\pi(q^{-1}-1)} \right)^{k/2} \frac{q^{\sum_{i=1}^k w_i}}{\sigma_{[W_1]_q} \prod_{j=2}^k \sigma_{[W_j]_q|(W_1,W_2,\ldots,W_{j-1})}}$$

$$\cdot \exp\left(\frac{1-q}{2\log q} \left(\left(\frac{[w_1]_q - \mu_{[W_1]_q}}{\sigma_{[W_1]_q}} \right)^2 + \sum_{j=2}^k \left(\frac{[w_j]_q - \mu_{[W_j]_q|(W_1,\ldots,W_{j-1})}}{\sigma_{[W_j]_q|(W_1,\ldots,W_{j-1})}} \right)^2 \right) \right),$$

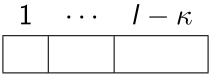
$$w_j \geq 0, j = 1, 2, \ldots, k, k \geq 2.$$

Example: Absorption Process (Kemp (1998), Charalambides (2012, 2016))

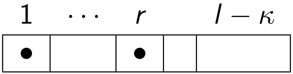


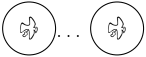


1$ Absorption

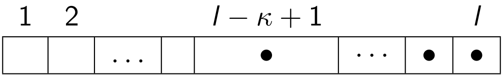


Propulsion

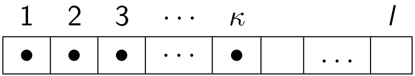




Escape



Propulsion without Absorption

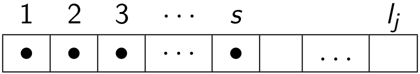


Conditional probability of an absorption of a batch of k particles, given that $j-1$ absorptions occur: $p_j = 1 - q^{l-\kappa j+1}$, $j = 1, 2, \dots$.

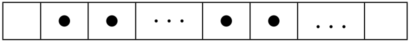
- $p_j = 1 - q^{\kappa(r-j+1)}$, $j = 1, 2, \ldots, [r]$, $l = (r+1)\kappa - 1$
- Y: number of absorbed batches of κ particles, when n batches are propelled into the chamber of l consecutive cells.
- Distribution of the r.v. Y: Absorption with parameter q^κ.
- The continuous limit of the probability function of Y, for $q = q(n)$, with $q(n) \to 1$, as $n \to \infty$, $q(n)^{n\kappa} = \Omega(1)$ and $r = O(n)$, is the deformed standardized Gaussian distribution, with $q = q^\kappa$.

Example: Multivariate Absorption Process (M.V., 2024)

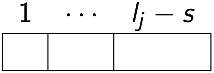
Chamber c_j with l_j lined cells, $j = 1, 2, \ldots, k$



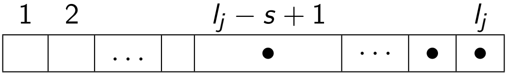
Fail of jth kind



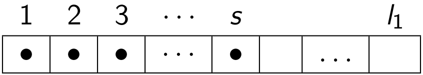
1$ Absorption: Success of the jth kind



... Escape from c_j chamber



Propulsion without Absorption



Conditional probability of an absorption of the jth kind of a batch of s particles, given that $i-1$ absorptions of the jth kind occur:
$p_{j,i} = 1 - q^{l_j - si + 1}$, $i = 1, 2, \ldots, l_j$, $j = 1, 2, \ldots, k$

- $p_{j,i} = 1 - q^{s(m_j - i + 1)}$, $i = 1, 2, \ldots, [m_j]$, $l_j = (m_j + 1)s - 1$, $j = 1, 2, \ldots, k$

- Y_j: number of absorbed batches of s particles of the jth kind, when n batches are propelled into the chamber c_j of l_j cells, $j = 1, 2, \ldots, k$

- Distribution of the r.v. $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_k)$: Multivariate absorption with parameter q^s.

- The continuous limit of the joint probability function of the random variables $Y_j, j = 1, 2, \ldots, k$ for $q = q(n)$, with $q(n) \to 1$, as $n \to \infty$, $q(n)^{ns} = \Omega(1)$ and $m_j = O(n), j = 1, 2, \ldots, k$ is the deformed multivariate standardized Gaussian distribution, with $q = q^s$.

- Charalambides, C.A.: Discrete q-Distributions, John Wiley Sons, New Jersey (2016)
- Charalambides, C. A.: q-Multinomial and negative q-multinomial distributions. Communications in Statistics - Theory and Methods 50, 5673-5898 (2021)
- Charalambides, C. A.: Multivariate q-Pólya and inverse q-Pólya distributions. Communications in Statistics - Theory and Methods 51, 4854-4876(2022)
- Kemp, A.: Heine–Euler extensions of the Poisson distribution. Comm. Statist. Theory Meth. 21, 571–588 (1992)
- Kemp, A.: Steady-state Markov chain models for certain q-confluent hypergeometric distributions. J. Statist. Plann. Inference 135, 107–120 (2005)
- Kemp, A.: Absorption sampling and the absorption distribution. J. Appl. Probab.35, 489–494 (1998)
- Kemp, A: A characterization of a distribution arising from absorption sampling. In Probability and Statistical Models and Applications, C. A. Charalambides, M.V. Koutras & N. Balakrishnan (eds.) 239–246. Chapman and Hall/CRC Press, Boca Raton, FL (2000)
- A. Kyriakoussis and M.G. Vamvakari, On terminating and non-terminating q-gauss hypergeometric series distributions and the associated q-orthogonal polynomials, Fundamenta Informaticae, 117, 229-248, 2012.
- Kyriakoussis, A., Vamvakari, M.G.: On a q-analogue of the Stirling formula and a continuous limiting behaviour of the q-Binomial distribution-Numerical calculations. Method. Comput. Appl. Probab. 15, 187–213 (2013)
- Kyriakoussis, A., Vamvakari, M.G.: Heine process as a q-analog of the Poisson process–waiting and interarrival times. Communications in Statistics - Theory and Methods 46, 4088-4102 (2017)
- Vamvakari, M.: On multivariate discrete q-distributions-A multivariate q-Cauchy's formula. Communications in Statistics - Theory and Methods 49, 6080-6095 (2020)
- Vamvakari, M.: Asymptotic behaviour of univariate and multivariate absorption distributions. In Randomness and Combinatorics, L. Ferrari & P. Massazza (eds.), RAIRO–Theoretical Informatics and Applications (to appear)

Thank you!!!