Bivariate Noncentral Pólya-Aeppli Distribution. An Application to Insurance Risk Theory

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In this talk a convolution of Poisson distribution and Pólya-Aeppli distribution is analyzed. The resulting distribution is called a Noncentral Pólya-Aeppli distribution.

Suppose that the nonnegative random variable Z_3 with a given parameter λ_3 has a Poisson distribution, i.e.,

$$P(Z_3 = i \mid \lambda_3) = \frac{\lambda_3^i}{i!} e^{-\lambda_3}, \ \ i = 0, 1, \dots . \qquad (1)$$

where $\lambda_3 > 0$ and we use the notation $Z_3 \sim Po(\lambda_3)$. It is known that $\mathrm{E}(Z_3) = \mathrm{Var}(Z_3) = \lambda_3$.

The mean and the variance of Z_1 are given by

$$\mathrm{E}(Z_1) = \frac{\lambda_1}{1-\rho}$$

and

$$\mathrm{Var}(Z_1) = \frac{\lambda_1(1+\rho)}{(1-\rho)^2}$$

The Fisher index is

$$FI(Z_1) = \frac{1+\rho}{1-\rho}$$

The Pólya-Aeppli distribution is a compound Poisson with geometric compounding distribution and it is analyzed in Minkova (2002) [4].

2.1. Probability generating function

The Noncentral Pólya-Aeppli distribution is a sum of independent Pólya-Aeppli distribution and Poisson distribution, i.e., $M = Z_1 + Z_3$, where Z_1 and Z_3 are independent random variables. Then the probability generating function (PGF) of the random variable M is given by

$$\Psi_N(s) = e^{-\lambda_1(1-\psi_1(s))} e^{-\lambda_3(1-s)}, \tag{2}$$

where

$$\psi_1(s) = \frac{(1-\rho)s}{1-\rho s} \tag{3}$$

is the PGF of the geometric distribution, $Ge_1(1-\rho)$.

Definition

The random variable M, defined by the PGF (2), has a Noncentral Pólya-Aeppli distribution (NPAD).

In 1979 Ong and Lee defined the Noncentral negative binomial distribution, [8]. It is the distribution of a sum of two independent variables. The first variable has a Negative binomial distribution and the second one is Pólya-Aeppli distributed. This was our motivation to name the resulting distribution Non-central Pólya-Aeppli distribution.

The probability mass function (PMF) of the Noncentral Pólya-Aeppli distribution is given by

$$P(M=i)$$

$$= \begin{cases} e^{-(\lambda_1+\lambda_3)}, \quad i=0 \\ e^{-(\lambda_1+\lambda_3)} \left[\sum_{j=1}^{i} \frac{(\lambda_3)^{i-j}}{(i-j)!} \sum_{k=1}^{j} \binom{j-1}{k-1} \frac{[\lambda_1(1-\rho)]^k}{k!} \rho^{j-k} + \frac{\lambda_3^i}{i!}\right], \\ \quad i=1,2,\dots, \end{cases} \tag{4}$$

Proposition

Denote by $p_i = P(M = i),\ i = 0, 1, \ldots.$
Then the PMF of the Univariate Noncentral Pólya-Aeppli distribution satisfies the following recursions:
$p_1 = [\lambda_1(1-\rho) + \lambda_3]p_0,$ *where* $p_0 = e^{-(\lambda_1+\lambda_3)}$ *and for* $i = 1, 2, \ldots$

$$p_{i+1} = \frac{[\lambda_1(1-\rho)+\lambda_3]}{i+1} p_i + \lambda_1(1-\rho)\sum_{j=0}^{i-1}\left[1 - \frac{j}{i+1}\right]\rho^{i-j} p_j$$

Moments

The mean and the variance of the Univariate Noncentral Pólya-Aeppli distribution are given by:

$$\mathrm{E}(M) = \left(\frac{\lambda_1}{1-\rho} + \lambda_3\right) \quad \text{and} \quad \mathrm{Var}(M) = \left(\lambda_1 \frac{1+\rho}{(1-\rho)^2} + \lambda_3\right)$$

For the Fisher index we obtain:

$$FI(M) = \frac{\lambda_3(1-\rho)^2 + \lambda_1(1+\rho)}{(1-\rho)[\lambda_3(1-\rho) + \lambda_1]}$$

It is easy to check that:

$$FI(M) = 1 + \frac{2\lambda_1\rho + \lambda_3(1-\rho)}{(1-\rho)[\lambda_3(1-\rho) + \lambda_1]},$$

i.e., NPAD is over-dispersed related to a Poisson distribution and

$$FI(M) = \frac{1+\rho}{1-\rho} - \frac{2\lambda_3\rho}{\lambda_3(1-\rho) + \lambda_1} < \frac{1+\rho}{1-\rho},$$

i.e., NPAD is under-dispersed related to Pólya-Aeppli distribution.

3. Bivariate Noncentral Pólya-Aeppli distribution

Let Z_1, Z_2 and Z_3 are independent variables. Suppose that $Z_i,\ i = 1, 2$ are Pólya-Aeppli distributed with parameters λ_i and ρ_i,, i.e, $Z_i \sim PA(\lambda_i, \rho_i)$. The random variable Z_3 has a Poisson distribution with parameters λ_3, i.e $Z_3 \sim Po(\lambda_3)$.
Let us also define the random variables:

$$M = Z_1 + Z_3 \ \text{ and } \ N = Z_2 + Z_3.$$

It is clear that:

$$M \sim NPA(\lambda_1 + \lambda_3, \rho_1) \ \text{ and } \ N \sim NPA(\lambda_2 + \lambda_3, \rho_2).$$

The Joint PGF of (M, N) is given by:

$$\psi(s_1, s_2) = e^{-\lambda_1(1-\frac{(1-\rho_1)s_1}{1-\rho_1 s_1})-\lambda_2(1-\frac{(1-\rho_2)s_2}{1-\rho_2 s_2}-\lambda_3(1-s_1 s_2)} \quad (5)$$

where

$$\psi_1(s_1) = \frac{(1-\rho_1)s_1}{1-\rho_1 s_1} \quad \text{and} \quad \psi_1(s_2) = \frac{(1-\rho_2)s_2}{1-\rho_2 s_2}$$

are the PGF-s of the compounding distribution given in (3).

Definition

The probability of (M, N) corresponding to (5) is referred to as a Bivariate Noncentral Pólya-Aeppli distribution, i.e., $BivNPA(\lambda_1, \lambda_2, \lambda_3, \rho_1, \rho_2)$.

Denote by $f(i,j) = P(M = i, N = j), \ i,j = 0,1,\ldots$ the Joint PMF of (M, N).
Then the following proposition gives an extension of the Panjer recursion formulas; see [9].

Proposition

The PMF of the BivPA$(\lambda_1, \lambda_2, \lambda_3, \rho_1, \rho_2)$ distribution satisfies the following recursions:

$$f(i+1,j) = \left[2\rho_1 + \tfrac{(1-\rho_1)\lambda_1 - 2\rho_1}{i+1}\right] f(i,j) - \rho_1^2(1 - \tfrac{2}{i+1}) f(i-1,j)$$

$$+ \tfrac{\lambda_3}{i+1}\left[f(i,j-1) - 2\rho_1 f(i-1,j-1) + \rho_1^2 f(i-2,j-1)\right],$$

$$f(i,j+1) = \left[2\rho_2 + \tfrac{(1-\rho_2)\lambda_2 - 2\rho_2}{j+1}\right] f(i,j) - \rho_2^2(1 - \tfrac{2}{j+1}) f(i,j-1)$$

$$+ \tfrac{\lambda_3}{j+1}\left[f(i-1,j) - 2\rho_2 f(i-1,j-1) + \rho_2^2 f(i-1,j-2)\right], \tag{6}$$

with $f(-1,0) = 0, f(0,-1) = 0$ *and* $i, j = 0, 1, \ldots$

Theorem

The PMF of the BivPA$(\lambda_1, \lambda_2, \lambda_3, \rho_1, \rho_2)$ distribution is given by:

$$f(i,0) = \sum_{m=1}^{i} \binom{i-1}{i-m} \frac{[\lambda_1(1-\rho_1)]^m}{m!} \rho_1^{i-m} f(0,0), \quad i = 1, 2, \dots,$$

$$f(0,j) = \sum_{l=1}^{j} \binom{j-1}{j-l} \frac{[\lambda_2(1-\rho_2)]^j}{j!} \rho_2^{j-l} f(0,0), \quad j = 1, 2, \dots,$$

$$f(i,j)$$

$$= \Bigg[\left(\sum_{m=1}^{i} \binom{i-1}{i-m} \frac{[\lambda_1(1-\rho_1)]^m}{m!} \rho_1^{i-m} \right) \left(\sum_{l=1}^{j} \binom{j-1}{j-l} \frac{[\lambda_2(1-\rho_2)]^l}{l!} \rho_2^{j-l} \right)$$

$$+ \sum_{k=1}^{i \wedge j} \frac{(\lambda_3)^k}{k!} \sum_{m=0}^{i-k} \binom{i-1}{i-k-m} \frac{[\lambda_1(1-\rho_1)]^m}{m!} \rho_1^{i-k-m}$$

$$\times \sum_{l=0}^{j-k} \binom{j-1}{j-k-l} \frac{[\lambda_2(1-\rho_2)]^l}{l!} \rho_2^{j-k-l} \Bigg] f(0,0), \quad i, j = 1, 2, \dots,$$

with $f(0,0) = e^{-(\lambda_1+\lambda_2+\lambda_3)}$.

Moments of the Bivariate Noncentral Pólya-Aeppli distribution

For the means of M and N we obtain:

$$\mathrm{E}(M) = \frac{\lambda_1}{1-\rho_1} + \lambda_3$$

and

$$\mathrm{E}(N) = \frac{\lambda_2}{1-\rho_2} + \lambda_3$$

The variances of M and N are:

$$\mathrm{Var}(M) = \lambda_3 + \frac{\lambda_1}{1-\rho_1}\left(1 + \frac{2\rho_1}{1-\rho_1}\right)$$

and

$$\mathrm{Var}(N) = \lambda_3 + \frac{\lambda_2}{1-\rho_2}\left(1 + \frac{2\rho_2}{1-\rho_2}\right).$$

The product moment of M and N is given by

$$\mathrm{E}(MN) = \lambda_3 + \left(\lambda_3 + \frac{\lambda_1}{1-\rho_1}\right)\left(\lambda_3 + \frac{\lambda_2}{1-\rho_2}\right)$$

which readily yield to the covariance between M and N to be

$$\mathrm{Cov}(M, N) = \lambda_3$$

The correlation coefficient:

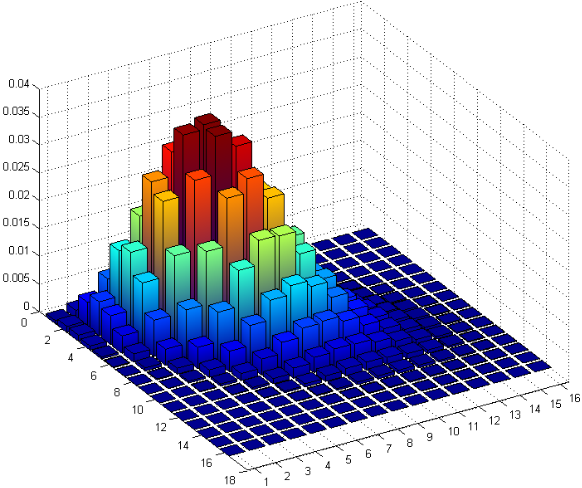
$$\mathrm{Corr}(M, N)$$

$$= \frac{\lambda_3 (1-\rho_1)(1-\rho_2)}{\sqrt{\left[\lambda_3 (1-\rho_1)^2 + \lambda_1 (1+\rho_1)\right] \left[\lambda_3 (1-\rho_2)^2 + \lambda_2 (1+\rho_2)\right]}}$$

Remark

When $\rho_1 = \rho_2 = 0$ the PMF of the Noncentral Pólya-Aeppli distribution coincides to the bivariate Poisson distribution.

$\rho_1 = \rho_2 = 0$



0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0
2
4
6
8
10
12
14
16
18
1
2
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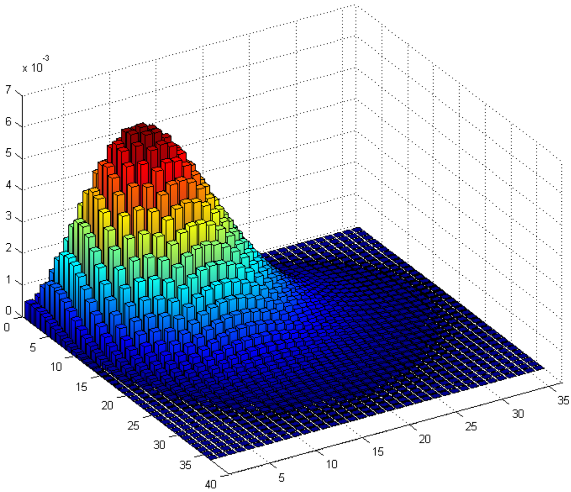
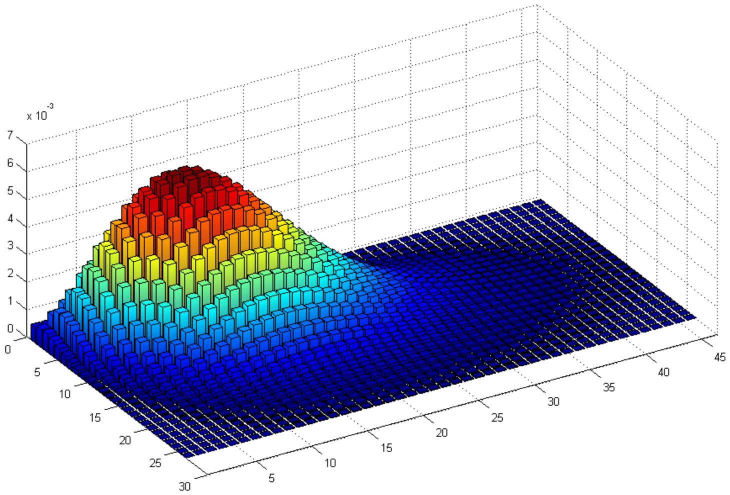
16

$\rho_1 = \rho_2 = 0.5$



$\rho_1 = 0.4, \ \rho_2 = 0.6$



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Thanks

Thank you for your kind attention !